Math 711: Lecture of November 2, 2005

We shall prove the following:

Theorem. *Let R be a finitely generated $\mathbb{N}$-graded ring over $R_0 = K$, a field, such that R is a direct summand, as an R-module, of a polynomial ring. Then the $\mathfrak{a}$-invariant of R is negative.*

The point is that it is known classically that both $K[X/r]$ and $K[X]/I_t(X)$ are such direct summands when $K = \mathbb{C}$ (or any field of characteristic 0). Note that this is false in positive characteristic. We can conclude that the $\mathfrak{a}$-invariaqnt of these rings is negative, and then the same holds in characteristic $p > 0$, as indicated earlier, because these rings arise by base change from Hodge algebras over $\mathbb{Z}$. This will complete our proof of the F-regularity of these rings in characteristic $p > 0$.

We first recall the relevant facts from classical invariant theory: what we need is proved in Hermann Weyl's book, *The Classical Groups.*

We recall that over the complex numbers $\mathbb{C}$ a linear algebraic group (i.e., a Zariski closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some n) is *reductive* if every representation is completely reducible. (A representation is assumed to be a directed union of representations on finite-dimensional vector spaces V over $\mathbb{C}$ such that action map $G \times V \to V$ is a morphism of algebraic sets over $\mathbb{C}$.) A linear algebraic group with the property that every representation is completely reducible (a direct sum of simple representations) is called *linearly reductive* in all characteristics. Note that the property of being reductive is defined differently over fields of characteristic p. The reductive groups over $\mathbb{C}$ are the complexifications of compact real Lie groups. They include finite groups, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$ and the orthogonal, special orthogonal, and symplectic groups. They are closed under taking products.

When a reductive group over $\mathbb{C}$ acts on a $\mathbb{C}$-algebra R, there is a canonical retraction $R \to R^G$, where R^G is the ring of invariants $\{r \in R : g(r) = r \text{ for all } g \in G\}$. Classically, this map was constructed by using Haar measure to average over the compact real Lie group of which G is the complexification. When G is finite, this is quite simple: the map takes $r \in R$ to

$$\frac{1}{|G|}\big(\sum_{g \in G} g(r)\big).$$

The map is R^G-linear. Such a map can be constructed algebraically by taking the complement of R^G to be the sum of all irreducibles in R that meet R^G in 0.

The key facts we need are the following, for which refer the reader to Weyl's book:

Theorem. *Let X be an $r \times s$ matrix over $\mathbb{C}$, where $1 \le r \le s$. Let $\mathrm{SL}(r, \mathbb{C})$ act on $\mathbb{C}[X]$ in such a way that the matrix $X \mapsto \alpha X$ for $\alpha \in SL(r, \mathbb{C})$. Then $R^G = \mathbb{C}[X/r]$.*

Of course, it is clear that the $r \times r$ minors of X are invariant: each $r \times r$ submatrix of X is multiplied by α, and its determinant is multiplied by $\det(\alpha) = 1$.

Theorem. *Let t, r, s be integers with $2 \leq t \leq \min\{r, s\}$. Let Y be an $r \times (t-1)$ matrix of indeterminates over $\mathbb{C}$ and let Z be a $(t-1) \times s$ matrix of indeterminates over $\mathbb{C}$. Let $G = \mathrm{GL}(t-1, \mathbb{C})$ act in such a way that if $\alpha \in G$ we have $Y \mapsto Y\alpha^{-1}$ and $Z \mapsto \alpha Z$. Then $K[Y, Z]^G = K[YZ]$. Moreover, $K[YZ] \cong K[X]/I_t(X)$.*

It is clear that the entries of YZ are invariant, since $Y\alpha^{-1}\alpha Y = YZ$, and it is clear that we have a surjection $K[X] \twoheadrightarrow K[YZ]$ that kills $I_t(X)$. Since we already know that $I_t(X)$ is prime, it is not difficult to prove that the induced surjection $K[X]/I_t(X) \twoheadrightarrow K[YZ]$ is an isomorphism.

Both of the theorems above are also correct over infinite fields of characteristic $p > 0$. But we do not have the Reynolds operator.

We shall also assume the following fact from tight closure theory:

Theorem. *In equal characteristic, a direct summand of a regular ring is Cohen-Macaulay.*

The result of Theorem 4.2 of the manuscript *Tight Closure theory and characteristic p methods*, which is available from the Web Page for Math 715, Fall 2002 suffices for applications. No restrictions on the regular ring are needed, however: the result does hold for all regular rings in equal characteristic. We will be using the result in equal characteristic 0, but only for affine algebras over a field, where it is readily deduced by reduction to characteristic p. See the manuscript *Tight Closure in Equal Characteristic 0* available from my Web Page.

The next statement, once proved, will show, when combined with the results from invariant theory mentioned above, that $\mathbb{C}[X/r]$ and $\mathbb{C}[X]/I_t(X)$ both have negative $\mathfrak{a}$-invariants, and will therefore complete the proof that the corresponding rings are F-regular in characteristic $p > 0$.

Theorem. *If R is a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, a field, such that R is a direct summand of a polynomial ring (as an R-module), then $\mathfrak{a}(R) < 0$.*

The result above holds no matter what the characteristic of the field K is.

We shall make use of Segre products in the proof. Let R and S be two finitely generated $\mathbb{N}$-graded K-algebras over K, where $R_0 = S_0 = K$. The *Segre product* $R \circledS_K S$ is defined as

$$\bigoplus_{i \in \mathbb{N}}^{\infty} R_i \otimes_K S_i,$$

which is a subring of

$$R \otimes_K S = \bigoplus_{i,j \in \mathbb{N}} R_i \otimes_K S_j.$$

Note that $R \otimes_K S$ has an $\mathbb{N} \times \mathbb{N}$-grading in which $[R \otimes_K S]_{i,j} = R_i \otimes_K S_j$ while $R \circledS_K S$ has an $\mathbb{N}$-grading with $[R \circledS_K S]_i = R_i \otimes S_i$.

Proposition. *Let R and S be two finitely generated $\mathbb{N}$-graded K-algebras over K, where $R_0 = S_0 = K$, of Krull dimension d and d', respectively.*

(a) *$R \circledS_K S$ is a direct summand of $R \otimes_K S$ as a module over $R \circledS_K S$.*

(b) *Let $x_1, \dots, x_d$ be a homogeneous system of parameters for R and let $y_1, \dots, y_d$ be a homogeneous system of parameters for S. Assume that all of the x_i and y_j have the same degree h. (Such systems always exist: they can be constructed by raising the elements in the two homogenous systems of parameters to suitable powers, so that each degree is the least common multiple of all of the original degrees.) Then $R \circledS_K S$ is module-finite over its subring $B = K[x_i y_j : 1 \le i \le d,\, 1 \le j \le d']$, where we identify each of R and S with its image in $R \otimes_K S$, and B is isomorphic with $K[Z]/I_2(Z)$, where Z is a $d \times d'$ matrix of indeterminates over K. Hence, the Krull dimension of $R \circledS_K S$ is $d + d' - 1$.*

Proof. For part (a), we note that one can take

$$\bigoplus_{j \neq k} R_j \otimes S_k$$

as an $(R \circledS_K S)$-module complement for $R \circledS_K S$ in $R \otimes_K S$: note that

$$(R_i \otimes_K S_i)(R_j \otimes_K S_k) \subseteq R_{i+j} \otimes_K S_{i+k},$$

and $i + j \neq i + k$ if $j \neq k$.

For part (b), it suffices to show that every element of $R \circledS_K S$ of the form $FG = F \otimes G$, where $F \in R$ and $G \in S$ are homogeneous of the same degree, is integral over $R \circledS_K S$. Replacing the element by its h th power, we may assume without loss of generality that $\deg F = hn = \deg G$. Choose a homogeneous equation of integral dependence for F on $K[x_1, \dots, x_d]$, and call the degree n. After multiplying through by $y_j^n N$, we see that each Fy_j^n is integral over B. Therfore, it suffices to show that FG is integral over $K[Fy_1^n, \dots, Fy_{d'}^n]$. Choose a homogeneous equation of integral dependence for G over $K[y_1^n, \dots, y_{d'}^n]$, say of degree N', and multiply through by $F^{N'}$ to get the desired result. The remaining statements are now clear. $\square$

The next two results will complete the proof that $\mathbb{N}$-graded direct summands of polynomial rings have a negative $\mathfrak{a}$-invariant.

Theorem. *Let R be a finitely generated algebra over $R_0 = K$, a field. Let $S = K[s, t]$ be a polynomial ring in two variables over R. If $R \circledS_K S$ is Cohen-Macaulay, then the $\mathfrak{a}$-invariant of R is negative.*

Proof. Choose a homogeneous system of parameters $x_1, \dots, x_d$ for R of the same degree h. If the $\mathfrak{a}$-invariant of R is ≥ 0 then we can choose a form $g \in R$ such that $\deg(g) \ge hd$ while $g \notin (x_1, \dots, x_d)R$. We shall obtain a contradiction by showing that there is a non-trivial relation on a homogeneous system of parameters for $R \circledS_K S$. Let $u = s^h$ and $v = t^h$. This is a homogeneous system of parameters for S, and so $R \circledS_K S$ is module-finite over $C = K[ux_1, \dots, ux_d, vx_1, \dots, vx_d]$, which is a ring of dimension $d + 1$. The elements

$$vx_1, vx_2 - ux_1, \dots, vx_{j+1} - ux_j, \dots, vx_d - ux_{d-1}, ux_d$$

form a homogeneous system of parameters for the ring C and, hence, for $R \circledS_K S$, which is module-finite over C. Since there are $d+1$ elements, it suffices to see that all of the elements ux_j, vx_j are nilpotent once we kill the ideal I generated by these elements.

We can arrange the $2d$ generators as the rows $ux_1, \ldots, ux_d$ and $vx_1, \ldots, vx_d$ of a $2 \times d$ matrix whose 2×2 minors vanish. In the quotient mod I, the image of the matrix looks like this:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{d-1} & 0 \\ 0 & a_1 & a_2 & \ldots & a_{d-2} & a_{d-1} \end{pmatrix},$$

where a_j is the image of ux_j and of vx_{j+1}. The vanishing of the leftmost 2×2 minor shows that a_1 is nilpotent. The vanishing of the 2×2 minor formed from the second and third columns then shows that a_2 is nilpotent mod a_1, and therefore nilpotent. A straightforward induction shows that each a_j is nilpotent modulo the ideal generated by its predecessors and therefore nilpotent. It follows that all elements of the maximal ideal of the quotient are nilpotent, as required.

To complete the argument, it will suffice to exhibit a non-trivial relation on the system of parameters just constructed. Choose σ to be any monomial in s and t whose degree is $\deg(g) - hd$. It is permissible for σ to be 1. The relation we seek is

$$g\sigma u^d(vx_1) + g\sigma u^{d-1}v(vx_2 - ux_1) + \cdots + g\sigma u^{d-j}v^j(vx_{j+1} - ux_j)$$
$$+ \cdots + g\sigma uv^{d-1}(vx_d - ux_{d-1}) - g\sigma v^d(ux_d) = 0.$$

Note that $g\sigma$ is a factor throughout, needed to make the coefficients on the parameters elements of the Segre product. One term from each summand cancels with one term from the next, a telescoping effect.

To complete the proof, it will suffice to show that the coefficient of ux^d, which is $g\sigma v^d$, is not in the ideal generated by the first d parameters. Suppose it were. Notice that there is an R-homomorphism $R[s,\, t] \to R$ such that $s \mapsto 1$ and $t \mapsto 1$. If $g\sigma v^d$ is in the ideal generated by the parameters, we may apply this homomorphism. Each of the parameters maps into the ideal $(x_1, \ldots, x_d)R$, while s, t and, hence, $v = t^h$, $u = s^h$ and σ map to 1. Since g maps to itself, we find that $g \in (x_1, \ldots, x_d)R$, a contradiction. $\square$

Proposition. *Let R be a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, a field, and $S = K[s,\, t]$, a polynomial ring in two variables. If R is a direct summand, as an R-module, of a polynomial ring, then $R \circledS_K S$ is Cohen-Macaulay.*

Proof. Suppose that $\rho : T \to R$ is an R-linear module retraction, where T is a polynomial ring over K. Applying $\_ \otimes_K K[s,t]$ gives an $R[s,t]$-linear retraction $T[s,t] \to R[s,t]$, and since $B = R \circledS_K K[s,t]$ is a direct summand of $R \otimes_K K[s,t] \cong R[s,t]$ as a B-module, it follows that B is a direct summand of the polynomial ring $T[s,t]$ as a B-module, and B is therefore Cohen-Macaulay. $\square$

We can now prove the Theorem that we stated at the outset.

Proof of the Theorem. By the Proposition above, if R is a direct summand of a polynomial ring, then $R \circledS_K [s,t]$ is Cohen-Macaulay, and so the $\mathfrak{a}$-invariant of R is negative by the preceding Theorem. $\square$

Hence:

Theorem. *The $\mathfrak{a}$-invariant of $K[X/r]$ and of $K[X]/I_t(X)$ is negative for all fields X.* $\square$

We can at last conclude:

Theorem. *The rings $K[X/r]$ and $K[X]/I_t(X)$ are F-regular for every field K of characteristic $p > 0$.* $\square$