# Math 711: Lecture of November 4, 2005

We extend the definition of local cohomolgy. Let $I$ be an ideal of a Noetherian ring $R$ and let $M$ be any $R$-module, not necessarily finitely generated. We define

$$H^j_I(M) = \varinjlim_t \mathrm{Ext}^j_R(R/I^t,\, M).$$

This is called the $i$ th *local cohomology module of $M$ with support in $I$.*

$$H^0_I(M) = \varinjlim_t \mathrm{Hom}_R(R/I^t,\, M)$$

which may be identified with $\bigcup_t \mathrm{Ann}_M I^t \subseteq M$. Every element of $H^j_I(M)$ is killed by a power of $I$. Evidently, if $M$ is injective then $H^j_I(M) = 0$ for $j \geq 1$. By a taking a direct limit over $t$ of long exact sequences for Ext, we see that if $0 \to M' \to M \to M'' \to 0$ is exact there is a functorial long exact sequence for local cohomology:

$$0 \to H^0_I(M') \to H^0_I(M) \to H^0_I(M'') \to \cdots \to H^j_I(M') \to H^j_I(M) \to H^j_I(M'') \to \cdots .$$

It follows that $H^j_I(\_\,)$ is the $j$ th right derived functor of $H^0_I(\_\,)$. In the definition we may use instead of the ideals $I^t$ any decreasing sequence of ideals cofinal with the powers of $I$. It follows that if $I$ and $J$ have the same radical, then $H^i_I(M) \cong H^i_J(M)$ for all $i$.

**Theorem.** *If $M$ is a finitely generated $R$-module over the Noetherian ring $R$, then $H^i_I(M) \neq 0$ for some $i$ if and only if $IM \neq M$, in which case the least integer $I$ such that $H^i_I(M) \neq 0$ is* $\mathrm{depth}_I M$.

*Proof.* $IM = M$ iff $I + \mathrm{Ann}_R M = R$, and every element of every $H^j_I(M)$ is killed by some power $I^N$ of $I$ and by $\mathrm{Ann}_R M$: their sum must be the unit ideal, and so all the local cohomology vanishes in this case.

Now suppose that $IM \neq M$, so that the depth $d$ is a well-defined integer in $\mathbb{N}$. We use induction on $d$. If $d = 0$, some nonzero element of $M$ is killed by $I$, and so $H^0_I(M) \neq 0$. If $d > 0$ choose an element $x \in I$ that is not a zerodivisor on $M$, and consider the long exact sequence for local cohomology arising from the short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0.$$

From the induction hypothesis, $H^j_I(M/xM) = 0$ for $j < d-1$ and $H^{d-1}_I(M/xM) \neq 0$. The long exact sequence therefore yields the injectivity of the map

$$H^{j+1}_I(M) \xrightarrow{x} H^{j+1}_I(M)$$

for $j < d-1$. But every element of $H^{j+1}_I(M)$ is killed by a power of $I$ and, in particular, by a power of $x$. This implies that $H^{j+1}_I(M) = 0$ for $j < d-1$. Since

$$H^{d-1}_I(M) \to H^{d-1}_I(M/xM) \to H^d_I(M)$$

is exact, $H_I^{d-1}(M)$, which we know is not 0, injects into $H_I^d(M)$. $\square$

Now let $\underline{f} = f_1, \dots, f_n$ generate an ideal with the same radical as $I$. Let $\mathcal{C}^\bullet(\underline{f}^\infty; R)$ denote the total tensor product of the complexes $0 \to R \to R_{f_j} \to 0$, which gives a complex of flat $R$-modules:

$$0 \to R \to \bigoplus_j R_{f_j} \to \bigoplus_{j_1<j_2} R_{f_{j_1}f_{j_2}} \to \cdots \to R_{f_1\cdots f_n} \to 0.$$

Let $\mathcal{C}^\bullet(\underline{f}^\infty; M) = \mathcal{C}^\bullet(\underline{f}^\infty; R) \otimes_R M$, which looks like this:

$$0 \to M \to \bigoplus_j M_{f_j} \to \bigoplus_{j_1<j_2} M_{f_{j_1}f_{j_2}} \to \cdots \to M_{f_1\cdots f_n} \to 0.$$

We temporarily denote the cohomology of this complex as $\mathcal{H}^\bullet_{\underline{f}}(M)$. It turns out to be the same, functorially, as $H^\bullet_I(M)$. We shall not give a complete argument here but we note several key points. First, $\mathcal{H}^0_{\underline{f}}(M) = \operatorname{Ker}(M \to \bigoplus_j M_{f_j})$ is the same as the submodule of $M$ consisting of all elements killed by a power of $f_j$ for every $j$, and this is easily seen to be the same as $H^0_I(M)$. Second, by tensoring a short exact sequence of modules $0 \to M' \to M \to M'' \to 0$ with the complex $\mathcal{C}^\bullet(\underline{f}^\infty; R)$ we get a short exact sequence of complexes. This leads to a functorial long exact sequence for $\mathcal{H}^\bullet_{\underline{f}}(\_)$. These two facts imply an isomorphism of the functors $H^\bullet_I(\_)$ and $\mathcal{H}^\bullet_{\underline{f}}(\_)$ provided that we can show that $\mathcal{H}^j_{\underline{f}}(M) = 0$ for $j \geq 1$ when $M$ is injective. We indicate how the argument goes, but we shall assume some basic facts about the structure of injective modules over Noetherian rings.

First note that if one has a map $R \to S$ and an $S$-module $M$, then if $\underline{g}$ is the image of $\underline{f}$ in $S$, we have $H^\bullet_{\underline{f}}(M) = H^\bullet_{\underline{g}}(M)$. This has an important consequence for local cohomology once we establish that the two theories are the same: see the Corollary below.

Every injective module over a Noetherian ring $R$ is a direct sum of injective hulls $E(R/P)$ for various primes $P$. $E(R/P)$ is the same as the injective hull of the reisdue class field of the local ring $R_P$. This, we may assume without loss of generality that $(R, m, K)$ is local and that $M$ is the injective hull of $K$. This enables to reduce to the case where $M$ has finite length over $R$, and then, using the long exact sequence, to the case where $M = K$, since $M$ has a finite filtration such that all the factors are $K$. Thus, we may assume that $M = K$. The complex $\mathcal{C}^\bullet(\underline{f}^\infty; R)$ is then a tensor product of complexs of the the form $0 \to R \to R \to 0$ and $0 \to R \to 0 \to 0$. If we have only the latter the complex has no terms in higher degree, while if there are some of the former we get a cohomogical Koszul complex $\mathcal{K}^\bullet(g_1, \dots, g_n; K)$ where at least one $g_j \neq 0$. But then $(g_1, \dots, g_n)K = K$ kills all the Koszul cohomology. Thus, we get vanishing of higher cohomology in either case. It follows that $\mathcal{H}^\bullet_{\underline{f}}(\_)$ and $H^\bullet_I(\_)$ are isomorphic functors, and we drop the first notation, except in the proof of the Corollary just below.

**Corollary.** *If $R \to S$ is a homomorphism of Noetherian rings, $M$ is an $S$-module, and ${}_RM$ denotes $M$ viewed as an $R$-module via restriction of scalars, then for every ideal $I$ of $R$, $H^\bullet_I({}_RM) \cong H^\bullet_{IS}(M)$.*

*Proof.* Let $f_1, \ldots, f_n$ generate $I$, and let $g_1, \ldots, g_n$ be the images of these elements in $S$: they generated $IS$. We have $H^\bullet_I({}_RM) \cong \mathcal{H}^\bullet_{\underline{f}}({}_RM) \cong \mathcal{H}^\bullet_{\underline{g}}(M) \cong H^\bullet_{IS}(M)$. $\square$

We note that the complex $0 \to R \to R_f \to 0$ is isomorphic to the direct limit of the cohomological Koszul complexes $\mathcal{K}^\bullet(f^t; R)$, where the maps between consecutive complexes are given by the identity on the degree 0 copy of $R$ and by multiplication by $f$ on the degree 1 copy of $R$ — note the commutativity of the diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "f^{t+1}"] & R \arrow[r] & 0 \\
0 \arrow[r] & R \arrow[r, "f^t"] \arrow[u, "\mathrm{id}"] & R \arrow[r] \arrow[u, "f"'] & 0
\end{tikzcd}.$$

Tensoring these Koszul complexes together as $f$ runs through $f_1, \ldots, f_n$, we see that

$$\mathcal{C}^\bullet(\underline{f}^\infty;\, M) = \varinjlim_t \mathcal{K}^\bullet(f_1^t, \ldots, f_n^t;\, M).$$

Hence, whenever $f_1, \ldots, f_n$ generate $I$ up to radicals, taking cohomology yields

$$H^\bullet_I(M) \cong \varinjlim_t H^\bullet(f_1^t, \ldots, f_n^t;\, M).$$

When $R$ is a Cohen-Macaulay ring of Krull dimension $d$ and $x_1, \ldots, x_d$ is a system of parameters, this yields $H^d_m(R) = \varinjlim_t R/(x_1^t, \ldots, x_d^t)R$, which was our definition of $H(R)$ in this case.

We next recall that when $(R, m, K)$ is a complete local ring and $E = E_R(K)$ is an injective hull of the residue class field (this means that $K \subseteq E$ and every nonzero submodule of $E$ meets $K$), there is duality between modules with ACC over $R$ and modules with DCC: if $M$ satsifies one of the chain conditions then $M^\vee = \operatorname{Hom}_R(M, E)$ satisfies the other, and the canonical map $M \to M^{\vee\vee}$ is an isomorphism in either case. In particular, when $R$ is complete local, the obvious map $R \to \operatorname{Hom}_R(E, E)$ is an isomorphism. An Artin local ring $R$ with a one-dimensional socle is injective as a module over itself, and, in this case, $E_R(K) = R$. If $R$ is Gorenstein and $x_1, \ldots, x_d$ is a system of parameters, one has that each $R_t = R/(x_1^t, \ldots, x_d^t)R$ is Artin with a one-dimensional socle, and one can show that in this case $E_R(K) \cong H^d_M(R)$. Knowing this, we can prove a local duality theorem for local cohomology when $R$ is Gorenstein.

**Theorem.** *Let $(R, m, K)$ be a Gorenstein local ring of Krull dimension $d$, and let $E = H^d_m(R)$, which is also an injective hull for $K$. Let $M$ be a finitely generated $R$-module. Then for every integer $j$, $H^j_m(M) = \operatorname{Ext}^{d-j}_R(M, R)^\vee$.*

*Proof.* Let $x_1, \ldots, x_d$ be a system of parameters for $R$. In the Cohen-Macaulay case, the local cohomology of $R$ vanishes for $i < d$, and so $\mathcal{C}^\bullet(\underline{x}^\infty; R)$, numbered backwards, is a flat resolution of $E$. Thus, $H^j_m(M) \cong \mathrm{Tor}^R_{d-j}(M,\, E)$. Let $G_\bullet$ be a projective resoultion of $M$ by finitely generated projective $R$-modules. Then $\mathrm{Ext}^{d-j}_R(M,\, R)^\vee \cong H^{d-j}\big(\mathrm{Hom}_R(G_\bullet,\, R),\, E\big)$ (since $E$ is injective, $\mathrm{Hom}_R(\_,\, E)$ commutes with the calculation of cohomology). The functor $\mathrm{Hom}_R\big(\mathrm{Hom}_R(\_,\, R),\, E\big)$ is isomorphic with the functor $\_ \otimes E$ when restricted to finitely generated projective modules $G$. To see this, observe that for every $G$ there is an $R$-bilinear map $G \times E \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(G,\, R),\, E\big)$ that sends $(g,\, u)$ (where $g \in G$ and $u \in E$) to the map whose value on $f : G \to R$ is $f(g)u$. This map is an isomorphism when $G = R$, and commutes with direct sum, so that it is also an isomorphism when $G$ is finitely generated and free, and, likewise, when $G$ is a direct summand of a finitely generated free module. But then $\mathrm{Ext}^{d-j}_R(M,\, R)^\vee \cong H_{d-j}(G_\bullet \otimes E) \cong \mathrm{Tor}^R_{d-j}(M,\, E)$, which is $\cong H^j_m(M)$, as already observed. $\square$

Note that if $R$ is a finitely generated $\mathbb{N}$-graded $K$-algebra of Krull dimension $d$ such that $R_0 = K$, a field, and $R = K[R_1]$, then with $X = \mathrm{Proj}(R)$, we have that

$$H^i(X,\, \mathcal{O}_X) \cong [H^{i+1}_m(R)]_0$$

for $i \geq 1$. This follows from the fact that if $f_1, \, \ldots \,, f_d$ is a homogeneous system of parameters for $R$ then $[H^\bullet_m(R)]_0$ is the cohomology of $[\mathcal{C}^\bullet(\underline{x}^\infty;\, R)]_0$, and this complex, with the first term dropped and degrees decreased by one, is the same as the Cech complex for the sheaf $\mathcal{O}_X$ with respect to the affine open cover of $X$ consisting of the $X_{f_j}$. Therefore, the fact that the $\mathfrak{a}$-invariant of $X$ is negative implies the vanishing of $H^{d-1}(X,\, \mathcal{O}_X)$.

If $X$ and $Y$ are Noetherian schemes over a field $K$ and $\mathcal{F}$, $\mathcal{G}$ are coherent sheaves on $X$ and $Y$ respectively, then one has the Künneth formula

$$H^k(X \times Y,\, \mathcal{F} \otimes_K \mathcal{G}) \cong \bigoplus_{i+j=k} H^i(X,\, \mathcal{F}) \otimes_K H^j(Y,\, \mathcal{G}).$$

This permits a geometric explanation of why Segre products with $R$ tend not to be Cohen-Macaulay in any instance where $H^{d-1}(X,\, \mathcal{O}_X) \neq 0$. For simplicity we only consider the case where $R = K[R_1]$ while $R_0 = K$. Then $R \,\textcircled{s}_K\, K[s,\, t]$ has dimension $d+1$. Let $Z = \mathrm{Proj}(R \,\textcircled{s}_K\, K[s,\, t]) \cong X \times_K \mathbb{P}^1_K$. If $R$ were Cohen-Macaulay then we would have $H^d_{\mathcal{M}}(R \,\textcircled{s}_K\, K[s,\, t]) = 0$, where $\mathcal{M}$ is the homogeneous maximal ideal of $R \,\textcircled{s}_K\, K[s,\, t]$, and so $H^{d-1}(Z,\, \mathcal{O}_Z) = 0$. Then $\mathcal{O}_Z = \mathcal{O}_X \otimes_K \mathcal{O}_Y$, and so, by the Künneth formula, $H^{d-1}(Z,\, \mathcal{O}_Z)$ is a direct sum of terms, one of which is $H^{d-1}(X,\, \mathcal{O}_X) \otimes_K H^0(\mathbb{P}^1_K,\, \mathcal{O}_{\mathbb{P}^1_K}) \cong H^{d-1}(X,\, \mathcal{O}_X) \otimes_K K \cong H^{d-1}(X,\, \mathcal{O}_X)$. Thus, $R \,\textcircled{s}_K\, K[s,\, t]$ cannot be Cohen-Macaulay if $H^{d-1}(X,\, \mathcal{O}_X) \neq 0$.

We next want to prove Reisner's result, stated earlier, concerning when face rings $K[\Delta]$ are Cohen-Macaulay. We need some preliminaries.

We first show that the result can be reduced to the case where the field $K$ has characteristic $p > 0$.

**Lemma.** *Let $R$ be a finitely generated $\mathbb{N}$-graded algebra over $R_0 = \mathbb{Z}$. Then $\mathbb{Q} \otimes_{\mathbb{Z}} R$ is Cohen-Macaulay if and only if $(\mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} R$ is Cohen-Macaulay for all but finitely many positive prime integers $p$.*

*Proof.* By clearing denominators, we may choose homogeneous elements $f_1, \ldots, f_d$ of positive degree such that the images of $f_1, \ldots, f_d$ form a homogeneous system of parameters for $\mathbb{Q} \otimes_{\mathbb{Z}} R$. After inverting one nonzero integer $a$ we may assume that $R_a$ and the Koszul homology $H_\bullet(f_1, \ldots, f_d;\, R_a) \cong H_\bullet(f_1, \ldots, f_d;\, R)_a$ consists of torsion-free $\mathbb{Z}_a$-modules: since the homogeneous components are finitely generated over $\mathbb{Z}_a$, all of these are $\mathbb{Z}_a$-free. For any prime $p$ not dividing $a$, $\mathbb{Z}_a/p\mathbb{Z}_a \cong \mathbb{Z}/p\mathbb{Z}$. Since the Koszul complex $\mathcal{K}(f_1, \ldots, f_d;\, R_a)$ and its homology consists of $\mathbb{Z}_a$-free modules, the calculation of homology commutes with tensoring with any $\mathbb{Z}_a$-module. Thus, $f_1, \ldots, f_d$ is a regular sequence in $\mathbb{Q} \otimes_{\mathbb{Z}} R$ if and only if its image in $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} R$ is is a regular sequence for all $p$ not dividing $a$. The fact that $f_1, \ldots, f_d$ is a homogeneous system of parameters implies that $\mathbb{Q} \otimes_{\mathbb{Z}} H_0(f_1, \ldots, f_d;\, R)$ is a finite-dimensional vector space over $\mathbb{Q}$. This implies that $H_0(f_1, \ldots, f_d;\, R_a)$ is a finite rank free $\mathbb{Z}_a$-module, and then $\mathbb{Z}/pZ \otimes_{\mathbb{Z}} H_0(f_1, \ldots, f_d; R_a) \cong H_0(f_1, \ldots, f_d;\, R/pR)$ will be finite-dimensional over $\mathbb{Z}/pZ$ for all $p$ not dividing $a$. $\square$

We need one more preliminary result.

**Lemma.** *Let $R$ be a finitely generated $\mathbb{N}$-graded algebra with $R_0 = K$, a field, and suppose that there are elements $x_j$ in the homogeneous maximal ideal $m$ of $R$ generating an ideal primary to $m$ such that each $R_{x_j}$ is Cohen-Macaulay. Suppose also that $R$ is of pure dimension $d$. The for all $j < d$, $H^j_m(R)$ has finite length.*

*Proof.* Let $T$ be a polynomial ring in $n$ variables such that $R = T/I$, where $I$ is homogeneous, and let $\mathcal{M}$ be the homogeneous maximal of $T$. Then all minimal primes of $I$ have height $n - d$. Moreover, $H^j_m(R) \cong H^j_m(R_m) \cong H^j_{\mathcal{M}}(R_m)$ is dual, working over $T_{\mathcal{M}}$, to $\mathrm{Ext}^{n-j}_{T_{\mathcal{M}}}(R_m,\, T_{\mathcal{Q}})$. It therefore suffices to prove that $W = \mathrm{Ext}^{n-j}_{T_{\mathcal{M}}}(R_m,\, T_{\mathcal{M}})$ has finite length. Let $\mathcal{Q}$ strictly contained in $\mathcal{M}$ be any prime ideal containing $I$ (localizing at a prime not containing $I$ kills $R$, and therefore certainly kills $W$), and let $P = \mathcal{Q}/I$. Then $W_{\mathcal{Q}} = \mathrm{Ext}^{n-j}_{T_{\mathcal{Q}}}(R_P,\, T_{\mathcal{Q}})$, and if $\operatorname{height} \mathcal{Q} = k$, this is dual over $T_{\mathcal{Q}}$ to $H^{k-(n-j)}_{\mathcal{Q}T_{\mathcal{Q}}}(R_P) = H^{k+j-n}_{PR_P}(R_P) = 0$: since $R_P$ is Cohen-Macaulay, its only nonvanishing local cohomology module with support in its maxmal ideal occurs when the exponent is $\dim(R_P) = k - (n - d) = k + d - n$, and $k + d - n > k + j - n$ since $j < d$. $\square$

We are now ready to prove Reisner's theorem. Note that, since we are working over a field, it does not matter whether we use homology or cohomology in the statement of the result: they are dual over $K$.

**Theorem (Reisner's criterion).** *Let $K$ be a field and let $\Delta$ be a finite simplicial complex over $K$ with vertices $x_1, \ldots, x_n$. Then $K[\Delta]$ is Cohen-Macaulay if and only if for every link $\Lambda$ of $\Delta$, the reduced simplicial cohomology with coefficients in $K$, $\widetilde{H}^i(\Lambda, K)$, vanishes for $0 \le i < \dim(\Lambda)$. Here, $\Delta$ itself is to be included among the choices for $\Lambda$, as the link of the empty simplex.*

*Proof.* Only the characteristic of the field matters in determining whether $K[\Delta]$ is Cohen-Macaulay. By applying the preceding Lemma to $\mathbb{Z}[\Delta]$, we may assume that the field has characteristic $p$. The Cohen-Macaulay hypothesis implies that $K[\Delta]$ has pure dimension. The same is true of Reisner's criterion: this comes down to verifying that $\Delta$ has pure dimension. We use induction. If the dimension of $\Delta$ is 0, this is clear. If it is positive, $\widetilde{H}^0(\Delta, K) = 0$, and so $\Delta$ is connected. If two facets have different dimensions, we can find a finite chain of facets from one to the other such that any two consecutive terms have a common vertex. Then two consecutive terms must have different dimensions, and this will also hold for the link of the vertex they have in common, contradicting the induction hypothesis, for the link also satisfies Reisner's criterion.

Now, for each variable $x_j$, $K[\Delta]_{x_j} \cong K[\Lambda_j][x_j,\, x_j^{-1}]$, where $\Lambda_j$ is the link of $x_j$, and $x_j$ is an indeterminate over $K[\Lambda_j]$. If $K[\Delta]$ is Cohen-Macaulay, it follows that all the rings $K[\Lambda_j]$ are, and we may assume inductively that all the rings $K[\Lambda_j]$ are Cohen-Macaulay if Reisner's criterion holds.

Thus, in proving the equivalence we may assume that $K[\Delta]$ is of pure dimension and that its local cohomology, $H^j_m(K[\Delta])$, has finite length $j < d = \dim(K[\Delta])$. We shall show that under these assumptions, the vanishing of $H^j_m(K[\Delta])$ for $j < d$ (which is equivalent to the Cohen-Macaulay property for $K[\Delta]$ ) is equivalent to the vanishing of $\widetilde{H}^j(\Delta, K)$ for $j < d + 1 = \dim(\Delta)$. There are two key points: one is that $K[\Delta]$ is F-split, and the other is that $H^j_m(K[\Delta])$ has a $\mathbb{Z}^n$-grading. We have already established that $K[\Delta]$ is F-split. The $\mathbb{Z}^n$-grading is a consequence of the fact that the local cohomology is the cohomology of the complex $\mathcal{C}^\bullet(\underline{x}^\infty;\, K[\Delta])$. It is easy to check that the action of Frobenius on this complex and its cohomology multiplies degrees in $\mathbb{Z}^n$ by $p$. This action is also injective, in all degrees, because $F : H^j_m(K[\Delta]) \to H^j_m(K[\Delta])$ may be thought of instead as the map $H^j_m(K[\Delta]) \to H^j_m(K[\Delta]^{1/p})$ induced by the inclusion $K[\Delta] \subseteq K[\Delta]^{1/p}$, and this inclusion is split. Since the local cohomology has finite length, no element can have a nonzero component except in degree $(0, 0, \ldots, 0)$: a nonzero component in any other degree will produce nonzero components in infinitely many degrees as we apply $F$ repeatedly, contradicting finite length. Therefore, $H^j_m(\Delta)$ vanishes for $j < d$ if and only if $H^j([\mathcal{C}^\bullet(\underline{x}^\infty;\, K[\Delta])]_{(0,\,0,\,\ldots\,,\,0)})$ vanishes for $j, d$.

But a typical term occurring as a direct summand in $\mathcal{C}^j(\underline{x};^\infty;\, K[\Delta])$ has the form $K[\Delta]_{x_{i_1}\cdots x_{i_j}}$ (where the $j$ subscripts are distinct), and this does not vanish if and only if $\{x_{i_1}, \ldots, x_{i_j}\}$ is a $(j-1)$-simplex of $\Delta$. Beyond that, the $(0, 0, \ldots, 0)$-homogeneous component of $K[\Delta]_{x_{i_1}\cdots x_{i_j}}$ is the $K$-vector space $K$ spanned by the identity element. Hence the complex $[\mathcal{C}^\bullet(\underline{x}^\infty;\, K[\Delta])]_{(0,\,0,\,\ldots\,,\,0)}$ is the same as the complex one uses to compute reduced simplicial cohomology, with degrees decreased by 1. The result now follows. $\square$