Math 711: Lecture of November 7, 2005

A simplicial complex is *constructible* if it is a simplex, or, recursively, the union of two contructible complexes of dimension n intersecting in a constructible complex of dimension $n-1$. It is easy to verify that every constructible simplicial complex Δ satisfies Reisner's criterion over every field K, by a straightforward induction that uses the Mayer-Vietoris sequence. The empty set is constructible, and, in dimension 0, every finite simplicial complex is constructible. In dimension 1 a simplicial complex is constructible if and only if it is connected.

In dimension 2, things get more complicated. A cylinder is not be constructible because the first cohomology group does not vanish. A triangulation of a real projective plane is not Cohen-Macaulay in characteristic 2, because the first cohomology (equivalently, homology) group won't vanish. If we think of the real projective plane as a 2-sphere with opposite points identified, it is clear that the fundamental group is $\mathbb{Z}_2$. On the other hand, a triangulation of a real projective plane gives a Cohen-Macaulay face ring in every characteristic except two. This means that the minimal resoluton of the corresponding defining ideal $I \subseteq K[x_1, \ldots, x_n]$ (where n is the number of vertices) generated by square-free monomials depends on the characteristic. Even its length depends on the characteristic, since it must be longer in characteristic 2: the length is $n - \mathrm{depth}_m K[x_1, \ldots, x_n]/I$ (m is the homogeneous maximal ideal), and the depth is equal to $\dim K[\Delta] = \dim(\Delta) - 1$ precisely when the ring is Cohen-Macaulay, and smaller otherwise.

We note that the Cohen-Macaulay property for $K[\Delta]$ only depends on the geometric realization $|\Delta|$ of Δ, not on the face ring, by a result of Munkres. We shall not give complete details, but we will give a sketch of the explanation. One can define the *reduced local cohomology* of the space X at x with coefficients in K, which we denote $\widetilde{H}^\bullet_x(X, K)$, as the direct limit over successively smaller open neighborhoods U of x of $\widetilde{H}^\bullet(Y_U, K)$ where Y_U is the quotient space obtained from X by collapsing $X - U$ to a point. One evidently need only take the limit over a family of choices for U that is cofinal in the set of all open neighborhoods of x. The local cohomology will, in general, depend on the cohomology theory being used, but when X is the geometric realization of Δ, one can choose the neighborhoods U such that the spaces Y_U have finite triangulations, and one can use simplicial cohomology throughout. We will return to this point momentarily. Note, for example, that when X is an n-manifold near x, we can choose arbitarily small neighborhoods U of x such that the closure D of U is an n-ball in n-space whose boundary is an $(n-1)$-sphere, and the spaces Y_U are n-spheres.

Then $K[\Delta]$ is Cohen-Macaulay if and only if for all $j < \dim(|\Delta|)$ and all $x \in |\Delta|$, $\widetilde{H}^j(\Delta) = \widetilde{H}^j_x(\Delta) = 0$. This turns out to be equivalent to Reisner's criterion: the key point is that the condition on vanishing of (topological) local cohomology is equivalent to the vanishing of the reduced cohomology of the various links Λ in degree $< \dim(\Lambda)$.

To explain why, we first recall that the *suspension* of the topological space X is the union of two cones over X identified only along X. For a simplicial complex Δ there is a simplicial version: a cone over Δ with one additional vertex y can be obtained by taking all simplices of Δ together will all sets that are the union of simplex of Δ and $\{y\}$. Call this $C_y(\Delta)$. The suspension is achieved as a simplicial complex, namely $C_y(\Delta) \cup C_z(\Delta)$, where y and z are two new vertices. Let $S(X)$ denote the suspension of X. An easy application of the Mayer-Vietoris theorem shows that $\widetilde{H}^i(X, K) = 0$ for all $i < \dim(X)$ iff $\widetilde{H}^i\big(S(X), K\big) = 0$ for all $i < \dim(X) + 1 = \dim\big(S(X)\big)$.

We want to understand the local cohomology of $|\Delta|$ as a function of the point x of $|\Delta|$ chosen. This turns out to depend only on the smallest simplex $\sigma \in \Delta$ such that x is an interior point of $|\sigma|$. The neighborhoods U of x can then be chosen arbitrarily small in such a way that the space Y_U is an iterated suspension of the link of σ, where the number of iterations is the same as the number of vertices of σ. The reduced cohomology then vanishes except in the highest dimension (the dimension of Δ) iff the same is true for the link Λ of σ, and this will complete the proof.

Recall that the *closed star* of x, where x is a vertex of Δ, consists of all simplices in Δ that contain x, and their subsets. The closed star is the cone with vertex x over the link of x. The *open star* of x is the geometric realization of the closed star with the geometric realization of the link of x subtracted. We want to verify the assertions of the preceding paragraph, first considering the case where x is a vertex of Δ. The open star U of x is an open neighborhood of x in $|\Delta|$ that contains arbitrarily small neighbhorhoods U_t homeomorphic with itself. To see this, note that the closed star is a union of rays emanating from x and terminating in a point $\lambda \in \Lambda$, the link of x. Use $t \in [0, 1]$ to parametrize the points of the ray linearly, with 0 corresponding to x and 1 to λ. Let Λ_t denote the set of all points parametrized by $t > 0$ on these rays: it is homeomorphic with Λ. Let U_t be the union of $\{x\}$ and the Λ_s for $0 < s < t$: it is homeomorphic with the open star. Given $U_{3t/2}$, with $t < 2/3$, the effect of collapsing $|\Delta| - U_t$ to a point is the same as the effect of collapsing $U_{3t/2} - U_t$ to a point y, and the resulting space can be thought of as the union of two cones over the link of x. One consists of x and the points in Λ_s for $0 < s \le 1/2$. The other consists of points Λ_s for $1/2 \le s < t$ together with the point z. Thus, Y_{U_t} is homeomorphic with the suspension of the link of x for all t. In fact, all the Y_{U_t} are homeomorphic with the space obtained by collapsing the link of x in the closed star of x. Hence, the reduced simplicial cohomology of the link vanishes in degree smaller than the dimension of the link if and only if the reduced simplicial local cohomology at x of $|\Delta|$ vanishes in degree smaller than the dimension of $|\Delta|$.

More generally, if x is interior to $|\sigma|$ with vertices $x_1, \ldots, x_k$, we get a closed neighborhood of x by taking the union of all $|\sigma'|$ for $\sigma' \supseteq \sigma$, which is an iterated cone over the geometric realization of the link Λ of σ. Call this iterated cone $C_\sigma(\Lambda)$, and also call it V. Let U be the set of points of V which, when written as a convex linear combination of the vertices of a simplex, involve all $x_1, \ldots, x_k$ with positive coefficients. Then U is an open neighborhood of x in $|\Delta|$. Let B be the union of the geometric realizaions of the simplices that do not contain at least one x_j. Then $U = V - B$, and Y_U may also be thought of as the result of collapsing $B = V - U \subseteq V$ to a single point. We want to show that x has arbitarily small open neighborhoods U_t such that Y_{U_t} is homeomorphic with Y_U. We also want to show that Y_U is homeomorphic with the result of iterated suspension of the link

Λ of x, where the iteration is performed k times.

To construct the neighborhoods U_t, simply note that for $t > 0$, if θ_t is the homothety centered at x that maps every ray from x into itself but multiplies distances from x by t, then, for t sufficiently small, if V_t, U_t and B_t denote the images of V, U, and B, respectively, then V_t is a closed neighborhood of x, and $U_t = V_t - B_t$ is an open neighborhood of x that may be made arbitrarily small. Moreover, the restriction of θ_t maps (V, U, B) homeomorphically onto (V_t, U_t, B_t). Thus, each Y_{U_t} is homeomorphic to Y_t for $0 < t \ll 1$.

It remains only to see that Y_U is the iterated suspension of the link Λ k-times. Note quite generally that when we take the cone over X and then collapse X to a point, we get the suspension of X. In our case, If σ_j denotes $\sigma - \{x_j\}$, we are collapsing the union of the $|C_{\sigma_j}(\Lambda)|$. For $j < k$, let $\tau_j = \sigma_j - \{x_k\}$. Let Z be the result of collapsing the union of the $|C_{\tau_j}(\Lambda)|$ for $j < k$ in $C_{\sigma_k}(\Lambda)$. By induction on k, we may identify Z with the $(k-1)$ st iterated suspension of $|\Lambda|$. Finally, Y_U may be identified with the result of collapsing Z in $C_{x_k}(Z)$, which is the suspension of Z, as required. $\square$

We next want to consider several interrelated conjectures and theorems for local rings. We shall eventually prove the various relations.

We begin by stating several of these.

(1) The direct summand conjecture. *Let R be a regular ring and S a module-finite extension of R. Then R is a direct summand of S as an R-module.*

(2) The monomial conjecture. *Let R be any local ring and $x_1, \ldots, x_d$ a system of parameters for R. Then for every nonnegative integer t, $x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})R$.*

(3) The canonical element conjecture. *Let (R, m, K) be a local ring of dimension d and $\underline{x} = x_1, \ldots, x_d$ a system of parameters for R. Let $\mathcal{G}_\bullet$ denote an exact sequence*

$$0 \to \operatorname{syz}^d(K) \to G_{d-1} \to \cdots \to G_0 \to K \to 0$$

where the G_j are free (so that this is a truncated free resolution of K). Here $\operatorname{syz}^d(K)$ is a d th module of syzygies of K. The canonical surjection $R/(x_1, \ldots, x_d) \to K$ lifts to a map $\phi_\bullet$ of complexes from the Koszul complex $\mathcal{K}_\bullet = \mathcal{K}_\bullet(x_1, \ldots, x_d;\, R)$ to the truncated resolution $\mathcal{G}_\bullet$:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \operatorname{syz}^d(K) & \longrightarrow & G_{d-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & K & \longrightarrow & 0 \\
 & & \phi_d \big\uparrow & & \phi_{d-1} \big\uparrow & & & & \phi_0 \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & R & \longrightarrow & \mathcal{K}_{d-1} & \longrightarrow & \cdots & \longrightarrow & R & \longrightarrow & R/(\underline{x})R & \longrightarrow & 0
\end{array}$$

where we have identified $\mathcal{K}_d \cong R \cong \mathcal{K}_0$. For any such lifiting, $\phi_d \neq 0$.

We shall eventually explain at length why this is called the canonical element conjecture.

Recall that if R is a domain, R^+ denotes the integral closure of R in an algebraic closure of its fraction field. Like the algebraic closure of a field, it is unique up to non-unique isomorphism.

(4) Nonvanishing of local cohomology of R^+. *If R is a complete local domain of Krull dimension d, then $H^d_m(R^+) \neq 0$.*

(5) Improved new intersection conjecture. *Let (R, m, K) be a local ring of dimension d and*

$$0 \to G_n \to \cdots \to G_0 \to 0$$

a complex of finitely generated free modules such that $H_0(G_\bullet)$ has a minimal generator that is killed by a power of m and $H_i(G_\bullet)$ has finite length for $i > 0$. Then $d \leq n$.

It turns out that the conjectures stated in (1) through (5) are all equivalent when R has positive prime characteristic p, and also when R/m has characteristic $p > 0$. Formally, they are also equivalent in equal characteristic 0, because all of them are true in that case. These statements are all known to be true in the equal characteristic case, and in mixed characteristic in dimension ≤ 3. The general case remains open. They are also known to imply several other important results, which we state below.

(6) The new intersection theorem. *Let (R, m, K) be a local ring of dimension d and*

$$0 \to G_n \to \cdots \to G_0 \to 0$$

a complex of finitely generated free modules such that $H_0(G_\bullet) \neq 0$ and $H_i(G_\bullet)$ has finite length for all i. Then $d \leq n$.

It is obvious that (5) $\Rightarrow$ (6).

(7) The intersection theorem. *Let (R, m, K) be a local ring and M, N finitely generated nonzero modules such that $M \otimes_R N$ has finite length. Then* $\dim(N) \leq \mathrm{pd}_R M$.

Of course, this is only interesting when $\mathrm{pd}_R M < \infty$.

To see that (6) $\Rightarrow$ (7), let $\widetilde{G}_\bullet$ be a minimal free resolution of M, so that its length is $n = \mathrm{pd}_R M$, and replace N by R/I, where $I = \mathrm{Ann}_R N$. Let $G_\bullet = (R/I) \otimes_R \widetilde{G}_\bullet$. Then the homology of this complex is killed by both I and $\mathrm{Ann}_R M$, which means that it is finite length, and so we can conclude that $\dim(N) = \dim(R/I) \leq n = \mathrm{pd}_R M$. $\square$

(8) Auslander's zerodivisor conjecture. *Let R be a local ring and $M \neq 0$ a finitely generated module of finite projective dimension. Let x be an element of R that is a zerodivisor on R. Then x is a zerodivisor on M.*

Under the same hypothesis, one can also state the contrapositive, that a nonzerodivisor on M is a nonzerodivisor in R. It follows that if $M \neq 0$ has finite projective dimension, then a regular sequence on M is a regular sequence in R.

(9) Bass's question. *Let R be a local ring that possesses a finitely generated nonzero module M of finite injective dimension. Then R is Cohen-Macaulay.*

It is not difficult to deduce both (8) and (9) from (7). (Note that if R is Cohen-Macaulay, it does possess a finitely generated R-module of finite injective dimension: if $\underline{x}$ is a system of parameters, the injective hull of K over $R/(\underline{x})R$ turns out to be such a module.) We should note that (6) has been proved in all characteristics by Paul Roberts, and so all of (6) through (9) are known in general. We also mention:

(10) The Evans-Griffith syzygy conjecture. *A k th module of syzygies of a finitely generated module over a regular local ring, if not free, has torsion-free rank at least k.*

This can be deduced from (5), and is known in equal characteristic but not, in general, in mixed characteristic. We shall discuss all of this in detail later.

We next want to explain the source of the name "canonical element conjecture" for (3).

We note that even for a fixed choice $\mathcal{G}_\bullet$ of truncated free resolution and a fixed system of parameters, the map ϕ_d is not unique: but it is unique up to homotopy of maps of complexes. This means that ϕ_d can be changed to any map of the form $\phi_d + h\delta_d$, where $\delta_d : \mathcal{K}_d \to \mathcal{K}_{d-1} \cong R^d$ has a $d \times 1$ matrix whose entries are, up to sign, the elements x_j and $h : \mathcal{K}_{d-1} \to \mathrm{syz}^d(K)$. Hence, $\phi_d(1)$ is unique in $\mathrm{syz}^d(K)/(x_1, \ldots, x_d)\mathrm{syz}^d(K)$. Therefore we may phrase the condition instead by saying the for every system of parameters, the image of $1 \in \mathcal{K}_d$ is nonzero $\mathrm{syz}^d(K)/(x_1, \ldots, x_d)\mathrm{syz}^d(K)$.

The condition becomes formally more difficult to satisfy as the system of parameters generates a smaller ideal. Suppose that A is $d \times d$ matrix such that $(y_1, \ldots, y_d) = (x_1, \ldots, x_d)A$, where $y_1, \ldots, y_d$ is also a system of paramters. As in the discussion of $H(M)$ in the Lecture of October 12, the exterior powers of A may be used to give a map of Koszul complexes $\mathcal{K}_\bullet(\underline{y};\, R) \to \mathcal{K}_\bullet(\underline{x};\, R)$. In degree d, the map is multiplication by $\det(A)$. This map of Koszul complexes may be composed with the map from $\mathcal{K}(\underline{x};\, R)$ to $\mathcal{G}_\bullet$ to obtain a map $\mathcal{K}_\bullet(\underline{y};\, R) \to \mathcal{G}_\bullet$, and the image of $1 \in R = \mathcal{K}_d(\underline{y};\, R)$ is $\det(A)$ times the image of $1 \subseteq \mathcal{K}_d(\underline{x};\, R)$. This means that we are obtaining a well-defined element η_R in $H^d_m\big(\mathrm{syz}^d(K)\big)$. (Even when the ring is not Cohen-Macaulay, for any R-module M, $H^d_m(M)$ may be viewed as

$$\varinjlim_{\underline{x}} M/(\underline{x})M$$

as $\underline{x}$ runs through all systems of parameters for R, and it suffices to consider

$$\varinjlim_{t} M/(x_1^t, \ldots, x_d^t)M$$

for a single system of parameters $\underline{x} = x_1, \ldots, x_d$, where the map

$$M/(x_1^t, \ldots, x_d^t)M \to M/(x_1^{t+1}, \ldots, x_d^{t+1})M$$

is induced by multiplication by $x_1 \cdots x_d$.) We refer to η_R as the *canonical element* in $H^d_m\big(\mathrm{syz}^d(K)\big)$.

One other point should be made: by passing to local cohomology, we have avoided dependence on the choice of system of parameters, and we have also avoided dependence on the choice of the map $\mathcal{K}(\underline{x};\, R)$ to $\mathcal{G}_\bullet$. However, the truncated resolution $\mathcal{G}_\bullet$ is not uniquely determined, and $\mathrm{syz}^d(K)$ is only determined up to adding a free direct summand and isomorphism. However, given two different free resolutions of K, one can lift the identity map on K to a map of complexes from each to the other. The compositions of these maps of complexes are homotopic to the respective identity maps. There are maps from the version of $\mathrm{syz}^d(K)$, call it S, arising from one resolution to the version, call it

S', arising from the other, and we get induced maps from each of $H^d_m(S)$, $H^d_m(S')$ to the other. Each map carries the canonical element in one local cohomology module to the canonical element in the other: this is immediate from the definition. In this sense, the canonical element η_R is independent on the choice of $\mathcal{G}_\bullet$, and whether it is zero or not is certainly independent of any of the choices made. Then the canonical element conjecture (3) can be stated as follows:

(3′) The canonical element conjecture. *For any local ring $(R,\, m,\, K)$ of Krull dimension d, the canonical element $\eta_d \in H^d_m\big(\mathrm{syz}^d(K)\big)$ is not zero.*

We shall eventually show that the canonical element can be obtained in a different way, as follows. The sequence

$$0 \to \mathrm{syz}^d(K) \to G_{d-1} \to \cdots \to G_0 \to K \to 0$$

can be viewed as an element of $\mathrm{Ext}^d_R(K,\, \mathrm{syz}^d(K))$ under the Yoneda definition of Ext. Call this element ϵ. If we think of $H^d_m\big(\mathrm{syz}^d(K)\big)$ as

$$\varinjlim_t \mathrm{Ext}^d_R\big(R/m^t,\, \mathrm{syz}^d(K)\big)$$

then ϵ resides in the $t = 1$ term of the direct limit system, and so has an image in $H^d_m\big(\mathrm{syz}^d(K)\big)$. This image is also the canonical element η_R.