Math 711: Lecture of November 9, 2005

We also note:

Proposition. *Let $(R,\, m,\, K)$ be a local ring of Krull dimension d.*

(a) *The canonical element η_R is nonzero if and only if for every map $\phi_\bullet$ from a Koszul complex $\mathcal{K}_\bullet = \mathcal{K}(\underline{x};\, R)$ on a system of parameters $\underline{x} = x_1, \ldots, x_d$ to a free resolution $G_\bullet$ of K lifting the canonical surjection $R/(\underline{x}) \twoheadrightarrow K$, the map $\phi_d : R \to G_d$ is nonzero.*

(b) *The the canonical element η_R is nonzero if and only if for every nonnegative left complex $G_\bullet$ of finitely generated free R-modules of length at least d with augmentation $M = H_0(G_\bullet) \neq 0$, for every system of parameters $\underline{x} = x_1, \ldots, x_d$ of R, and for every map $\phi_\bullet$ of the Koszul complex $\mathcal{K}_\bullet = \mathcal{K}(\underline{x}; R)$ to $G_\bullet$ such that the image in M of $1 \in R = \mathcal{K}_0$ is a minimal generator of M, the map $\phi_d : R \to G_d$ is nonzero.*

Proof. Part (a) follows from the fact that the near the dth spots the map of complexes looks likes this:

$$\begin{array}{ccccccc}
\cdots & \to & G_d & \twoheadrightarrow \mathrm{syz}^d(K) \hookrightarrow & G_{d-1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \to & \mathcal{K}_d & \xrightarrow{\hspace{2cm}} & \mathcal{K}_{d-1} & \to & \cdots
\end{array}$$

If the map ϕ_d is 0, we may use the composition of ϕ_d and the surjection $G_d \twoheadrightarrow \mathrm{syz}^d(K)$ to give a map to the truncated resolution which is 0 at the dth spot. On the other hand, given a map to the truncated resolution such that the map $\mathcal{K}_d \to \mathrm{syz}^d(K)$ is 0, we can obviously use it to give a map to the free resolution in which the map $\mathcal{K} \to G_d$ is 0.

The condition in part (b) is obviously sufficient from part (a), even if $G_\bullet$ is restricted to be a free resolution of K. We shall show that (a) implies (b). In the situation of (b), the image of $1 \in R = \mathcal{K}_0$ in M/mM is nonzero, and so we can choose a surjection $M \twoheadrightarrow K$ so that this element maps to $1 \in K$. This surjection $M \to K$ lifts to a map from $G_\bullet$ to a free resolution $G'_\bullet$ of K. The composition of the given map $\mathcal{K}_\bullet \to G_\bullet$ with this map $G_\bullet \to G'_\bullet$ contradicts (a). $\square$

As a corollary of this result we have:

Proposition. *Let $(R,\, m,\, K) \to (S,\, n,\, L)$ be a map a local map of local rings of the same dimension such that mS is primary to n. If $\eta_S \neq 0$, then $\eta_R \neq 0$.*

Proof. The hypotheses imply that a system of parameters for R maps to a system of parameters for S. Suppose that one has a map from the Koszul complex $\mathcal{K}_\bullet$ of a system of parameters of R to a free resolution of K over R such that the map at the dth spot is 0. Simply apply $S \otimes_R \_$ to contradict part (b) of the preceding Proposition over S. $\square$

Corollary. *Let $(R,\, m,\, K)$ be a complete local ring of Krull dimension d such that $\eta_R = 0$, let $\widehat{R}$ denote the completion of R, let S be the quotient of $\widehat{R}$ by a minimal prime such that* $\dim(S) = d$*, and let T be the normalization of S, which is a complete normal local domain. If $\eta_R = 0$, then $\eta_{\widehat{R}} = 0$, $\eta_S = 0$, and $\eta_T = 0$.*

Hence, if the canonical element conjecture holds for complete normal local domains, then it holds for all local rings.

Proof. All of the maps $R \to \widehat{R} \to S \to T$ satisfy the conditions of the preceding Proposition. $\square$

We shall now develop a variant of the Koszul complex which is always acyclic when the residual characteristic is $p > 0$, and which we shall use to show that a family of special cases of the direct summand conjecture implies the canonical element conjecture. In fact, we shall show that this family of cases, the canonical element conjecture, the monomial conjecture, and the direct summand conjecture are all equivalent.

Until further notice, let $(R,\, m,\, K)$ denote a complete local domain of residual characteristic $p > 0$, and let S denote either R^+ (which can be used either in the mixed characteristic case or in the case where R has characteristic $p > 0$) or R^∞ (which can be used in the characteristic $p > 0$ case). The property of S that we really need is that it be a quasilocal domain that is integral over R, and that is closed under extraction of all p^e th roots. Moreover, in the mixed characteristic case, it is convenient to assume that S contains all p^e th roots of unity.

If $x \in S$ is not zero, we write (x^∞) for the ideal

$$\bigcup_{e \in \mathbb{N}} x^{1/p^e} S.$$

Since S contains all p^e th roots of unity, it does not matter which choice of x^{1/p^e} we make in describing the principal ideal $x^{1/p^e}S$. Evidently, (x^∞) is an increasing union of principal ideals of S, which are free S-modules, and so every (x^∞) is a flat ideal of S. We note the following properties of these ideals.

Proposition. *Let notation be as in the preceding paragraph.*

(a) *For all nonzero $x,\, y \in S$,*

$$(x^\infty) \cap (y^\infty) = (x^\infty)(y^\infty) = \big((xy)^\infty\big),$$

and these are all isomorphic with $(x^\infty) \otimes_S (y^\infty)$, a flat ideal of S.

(b) *For any finite set $x_1, \, \dots \, , x_k$ of nonzero elements of S,*

$$\prod_{j=1}^{k} ({x_j}^\infty) = \bigcap_{j=1}^{k} ({x_j}^\infty) = ((\prod_{j=1}^{k} x_j)^\infty),$$

and all of these are isomorphic with $({x_1}^\infty) \otimes_S \cdots \otimes_S ({x_k}^\infty)$, a flat ideal of S.

(c) *For any nonzero x, (x^∞) is a radical ideal of S.*

Proof. First note that if $u \in (x^\infty)$, then $u^{1/p^e} \in (x^\infty)$ for all e, since $(sx^{1/p^n})^{1/p^e}$ has the form $s^{1/p^e}x^{1/p^{n+e}}$. Part (c) is immediate, since if $u^N \in (x^\infty)$, by increasing N if necessary we may assume that it has the form p^e.

For part (a), if $u \in (x^\infty) \cap (y^\infty)$ we also have that $u^{1/p} \in (x^\infty) \cap (y^\infty)$, and so $u = u^{1/p}(u^{1/p})^{p-1} \in (x^\infty)(y^\infty)$, and it is easy to see that this is the same as $\big((xy)^\infty\big)$.

Quite generally if I is a flat ideal of R and J is any ideal, then we may apply $I \otimes_R \_$ to the inclusion $0 \to J \hookrightarrow R$ to obatin $0 \to I \otimes_R J \hookrightarrow J$, and the image of this injection is evidently IJ, so that $I \otimes_R J \cong IJ$. When J is flat as well, $I \otimes_R J$ is flat. (b) follows at once by a straightforward induction. □

In characteristic p, we have a stronger result:

Proposition. *Let T be a ring of positive prime characteristic p such that the Frobenius endomorphism is an automorphism. This means that T is reduced and that every element has unique p^e th roots for all e. For every element $x \in T$, let*

$$(x^\infty) = \bigcup_{e \in \mathbb{N}} x^{1/p^e}.$$

Then an ideal of T is radical if and only if it is a (possibly) infinite sum of ideals of the form (x^∞). Moreover for any two radical ideals I and J, $I \cap J = IJ$, and the corresponding fact holds for finitely many radical ideals.

Proof. If a radical ideal contains x, it obviously must contain x^{1/p^e} for all e, and so contains (x^∞). To see that a sum of ideals (x^∞) is radical, observe that, by a direct limit argument, it suffices to see this for finitely may. By induction, it then suffices to prove this for two ideals. But if $u^N \in (x, y)T$, we can replace N by a larger integer of the form p^e, and, if $u^{p^e} = t_1 x + t_2 y$, then $u = t_1^{1/p^e} x^{1/p^e} + t_2^{1/p^e} y^{1/p^e} \in (x^\infty) + (y^\infty)$, so that $(x^\infty) + (y^\infty) = \mathrm{Rad}\,(x,\, y)$ is radical. □

The usual Koszul complex of elements $x_1, \, \dots \, , x_d$ in S may be thought of as the tensor product over S of the d complexes $0 \longrightarrow S \xrightarrow{x_j} S \to 0$. When each of the x_j is not a zerodivisor, we can instead think of these complexes as $0 \to x_j S \hookrightarrow S \to 0$, and the k th term of the tensor product can be thought of as the direct sum of all the k-fold tensor products

$$x_{j_1} S \otimes_S \cdots \otimes_S x_{j_k} S$$

where $j_1 < \cdots < j_k$. The displayed ideal may be identified with the prinicipal ideal $x_{j_1} \cdots x_{j_k} S$.

It is therefore natural to consider a similar construction with $x_j S$ replaced by $({x_j}^\infty)$: we tensor together the d complexes $0 \to ({x_j}^\infty) \hookrightarrow S \to 0$. This produces a complex in which the k th term may be thought of as

$$({x_{j_1}}^\infty) \otimes_S \cdots \otimes_S ({x_{j_k}}^\infty),$$

and, again, the tensor product may be replaced by the product (or, in this case, the intersection) of the ideals. Note that every module occurring is a direct sum of flat ideals, and therefore every term is flat. We denote this complex $\mathcal{K}_\bullet(({x_1}^\infty), \, \dots \, , ({x_d}^\infty), \, S)$. We shall see that, unlike ordinary Koszul complexes, these complexes are always acyclic! We shall likewise see that the residue class field of S has finite Tor dimension over S.