Math 711: Lecture of November 11, 2005

The following result is a key step in the proof of the acyclicity of the analogues of Koszul complexes $\mathcal{K}_\bullet((x_1{}^\infty), \ldots, (x_d{}^\infty); S)$ that we introduced in the Lecture of November 9.

Lemma. *Let (R, m, K) be a complete local domain of residual characteristic $p > 0$ and let S be either R^+ or R^∞ in the characteristic $p > 0$ case. Let $x_1, \ldots, x_d$ be elements of S, and, in the mixed characteristic case, assume that $x_1 = p$. Then for every k, $1 \leq k < d$,*

$$(x_{k+1}{}^\infty) \cap \bigl((x_1{}^\infty) + \cdots + (x_k{}^\infty)\bigr) = (x_{k+1}{}^\infty)(x_1{}^\infty) + \cdots + (x_{k+1}{}^\infty)(x_k{}^\infty).$$

Proof. In characteristic $p > 0$ we have that F is an automorphism of R. Since the two ideals being intersected on the left hand side are both radical, the intersection is the same as their product.

Now assume that R has mixed characteristic $p > 0$. We are assuming that $x_1 = p$. It follows that F is an automorphism of the ring $S/(p^\infty)$. Let v be an element of left hand side. Then, by using the characteristic p result applied to the ring $S/(p^\infty)$, we have that v is contained in

$$(p^\infty) + (x_{k+1}{}^\infty)(x_2{}^\infty) + \cdots + (x_{k+1}{}^\infty)(x_k{}^\infty).$$

Let $u \in (p^\infty)$ be the first term needed in expressing v as an element of the displayed sum. Since v and all the other terms are in $(x_{k+1}{}^\infty)$, it follows that $u \in (x_{k+1}{}^\infty)$ as well, and so

$$u \in (x_{k+1}{}^\infty) \cap (x_1{}^\infty) = (x_{k+1}{}^\infty)(x_1{}^\infty),$$

and the required result follows. $\square$

Theorem. *Let (R, m, K) be a complete local domain of residual characteristic $p > 0$ and let S be either R^+ or R^∞ in the characteristic $p > 0$ case. Let $x_1, \ldots, x_d$ be elements of S, and, in the mixed characteristic case, assume that one of these elements is p. Then the complex*

$$\mathcal{K}_\bullet((x_1{}^\infty), \ldots, (x_d{}^\infty); S)$$

is acyclic.

Proof. Permuting the $(x_j{}^\infty)$ does not affect the problem, and so in the mixed characteristic case we may assume that $x_1 = p$ without loss of generality. We prove by induction on k, where $1 \leq k \leq d$, that the complex

$$\mathcal{K}_\bullet((x_1{}^\infty), \ldots, (x_k{}^\infty); S)$$

is acyclic. The case where $k = 1$ is trivial. Therefore we may assume that

$$\mathcal{K}_\bullet = \mathcal{K}_\bullet((x_1{}^\infty), \ldots, (x_k{}^\infty); S)$$

is acyclic for some k with with $1 \le k < d$, and we must show that the total complex $\mathcal{T}_\bullet$ of

$$\mathcal{K}_\bullet \otimes_S (0 \to ({x_{k+1}}^\infty) \to S \to 0)$$

is acyclic. Since $\mathcal{T}_\bullet$ is a mapping cone, and has $\mathcal{K}_\bullet \otimes_S S = \mathcal{K}_\bullet$ as a subcomplex and $\mathcal{K}_\bullet \otimes_S ({x_{k+1}}^\infty)$ as a quotient complex, but with degreee decreased by one, there is a long exact sequence of homology:

$$\cdots H_j(\mathcal{K}_\bullet) \to H_j(\mathcal{T}_\bullet) \to H_{j-1}(\mathcal{K}_\bullet) \otimes_S ({x_{k+1}}^\infty) \to \cdots,$$

using that $({x_{k+1}}^\infty)$ is flat. If $j \ge 2$ it is immediate from the induction hypothesis that $H_j(\mathcal{T}_\bullet) = 0$, as required. The only issue is the vanishing of $H_1(\mathcal{T}_\bullet)$, and the relevant part of the long exact sequence is:

$$0 \to H_1(\mathcal{T}_\bullet) \to H_0(\mathcal{K}_\bullet) \otimes_S ({x_{k+1}}^\infty) \to H_0(\mathcal{T}_\bullet)$$

where the first zero is a consequence of the vanishing of $H_1(\mathcal{K}_\bullet)$. The vanishing of $H_1(\mathcal{T}_\bullet)$ will then follow from the injectivity of the rightmost map, which we can make more explicit as

$$\Big(S/\big(({x_1}^\infty) + \cdots ({x_k}^\infty)\big)\Big) \otimes_S ({x_{k+1}}^\infty) \to S/\big(({x_1}^\infty) + \cdots + ({x_k}^\infty)\big)$$

or

$$({x_{k+1}}^\infty)/\Big(({x_{k+1}}^\infty)\big(({x_1}^\infty) + \cdots ({x_k}^\infty)\big)\Big) \to S/\big(({x_1}^\infty) + \cdots + ({x_k}^\infty)\big).$$

The kernel of this map is clearly

$$\frac{({x_{k+1}}^\infty) \cap \big(({x_1}^\infty) + \cdots ({x_k}^\infty)\big)}{({x_{k+1}}^\infty)\big(({x_1}^\infty) + \cdots ({x_k}^\infty)\big)},$$

and the result now follows from the preceding Lemma. □

Corollary. *Let $(R,\, m,\, K)$ be a complete local domain of residual characteristic $p > 0$ and let S be either R^+ or R^∞ in the characteristic $p > 0$ case. Let $x_1, \, \ldots\, , x_d$ be a system of parameters for R, and, in the mixed characteristic case, assume that one of these elements is p. Then the complex $\mathcal{K}_\bullet(({x_1}^\infty),\, \ldots\, ,\, ({x_d}^\infty);\, S)$ is acyclic and therefore gives a flat resolution of the residue class field of S.*

Proof. The only point that needs to be checked is that $({x_1}^\infty) + \cdots + ({x_d}^\infty)$ is the maximal ideal of S. Since S is a directed union of module-finite domain extensions of R, each of which is local, it is clear $x_1, \, \ldots\, , x_d$ is a system of parameters in each of these rings, and therefore that every element of the maximal ideal of S is nilpotent modulo $(x_1, \, \ldots\, , x_d)S$. Hence, it suffices to see that $({x_1}^\infty) + \cdots + ({x_d}^\infty)$ is a radical ideal. But one of the x_j is p, (p^∞) is radical in S, and modulo (p^∞) we have a sum of ideals of the form (x^∞) in the reduced ring $S/(p^\infty)$. We know that such a sum is radical. □

It is remarkable that in a certain sense R^+ and R^∞ are like regular rings, in that in each the residue class field has finite Tor dimension.

Before starting the next theorem, we first make some very general comments about the direct summand conjecture. Let R be regular and let S be a module-finite extension. Then $R \to S$ splits if and only if the map

$$\mathrm{Hom}_R(S,\, R) \to \mathrm{Hom}_R(R,\, R)$$

is onto. Call the cokernel C. If $C \neq 0$, we can localize at a prime ideal in its support, and get a counterexample over a local ring. Thus, the direct summand conjecture reduces to the case where R is a regular local ring. Moreover, if R is local we may apply $\widehat{R} \otimes_R \_$ and so reduce to the case where R is a complete regular local ring. Moreover, we may assume without loss of generality that S is a domain: if not, we can kill a minimal prime P of S disjoint from $R - \{0\}$. The composition of $S \to S/P$ with a splitting of $R \to S/P$ provides a splitting of $R \to S$.

We can now prove:

Theorem. *For local rings of residual characteristic $p > 0$ and Krull dimension d the following statements are equivalent:*
(1) *The canonical element conjecture holds.*
(2) *The monomial conjecture holds.*
(3) *The direct summand conjecture holds.*
(4) *The direct summand conjecture holds over complete regular local rings which, in mixed characteristic, have the form $W[[x_2, \ldots, x_d]]$, where W has the form $V[p^{1/p^e}]$, with V a mixed characteristic p discrete valuation ring such that the maximal ideal is pV.*

Proof. We shall show that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1). The most interesting point is the implication (4) ⇒ (1), which is the only implication for which we use residual characteistic p.

To see that (1) ⇒ (4), suppose that for a local ring R one has a system of parameters $x_1, \ldots, x_d$ and a positive integer t such that $(x_1 \cdots x_d)^t \in (x_1^{t+1}, \ldots, x_d^{t+1})$. This implies that under the standard map of Koszul complexes

$$\mathcal{K}_\bullet(x_1^{t+1}, \ldots, x_d^{t+1};\, R) \to \mathcal{K}_\bullet(x_1, \ldots, x_d;\, R),$$

the image of $1 \in R = \mathcal{K}_d(x_1^{t+1}, \ldots, x_d^{t+1};\, R)$, which is $(x_1 \cdots x_d)^t$, is in

$$(x_1^{t+1}, \ldots, x_d^{t+1})\mathcal{K}_d(x_1, \ldots, x_d;\, R),$$

and so the last map may be altered by a homotopy to be 0, contradicting part (b) of the first Proposition of the Lecture of November 9.

We next want to see that (2) ⇒ (3). It suffices to consider the case of a complete local regular ring $R \subseteq S$. Let $E = H^d_m(R)$, which in this case is the injective hull of the residue class field. We claim that it suffices to prove that the map $R \to S$ remains injective when we apply $E \otimes_R \_$, i.e., that if $E \to S \otimes_R E$ is injective then

$$\mathrm{Hom}_R(S,\, R) \to \mathrm{Hom}_R(R,\, R) = R$$

is onto. But if the map is injective we may apply the exact functor $\mathrm{Hom}_R(\_, E)$ to get that

$$\mathrm{Hom}_R(S \otimes_R E, E) \to \mathrm{Hom}_R(E, E)$$

is surjective. By the adjointness of tensor and Hom, the term on the left can be identified with

$$\mathrm{Hom}_R\big(S, \mathrm{Hom}_R(E, E)\big),$$

and by Matlis duality, $\mathrm{Hom}_R(E, E) \cong R$, yielding that $\mathrm{Hom}_R(S, R) \to R$ is surjective. Since

$$E = H^d_m(R) = \varinjlim_t R/(x_1^t, \dots, x_d^t)R,$$

where $x_1, \dots, x_d$ is a regular system of parameters in R, it suffices to show that

$$R/(x_1^t, \dots, x_d^t)R \to S \otimes_R R/(x_1^t, \dots, x_d^t)R$$

is injective for all t. In $R/(x_1^t, \dots, x_d^t)R$, the socle is spanned by the image of $(x_1 \cdots x_d)^{t-1}$, and since every nonzero submodule of $R/(x_1^t, \dots, x_d^t)R$ contains the socle, if any element maps to 0, then the image of $(x_1 \cdots x_d)^{t-1}$ maps to 0. But this happens precisely when $(x_1 \cdots x_d)^{t-1} \in (x_1^t, \dots, x_d^t)S$, and so if R is not a direct summand of S, then the monomial conjecture fails for S.

The implication (3) $\Rightarrow$ (4) is obvious, and so it only remains to prove that (4) $\Rightarrow$ (1). Suppose that we can map the Koszul complex on a system of parameters generating the ideal I to a free resolution $G_\bullet$ of K lifting the surjection $R/I \to K$ in a such a way that d th map is 0, giving a counterexample to the canonical element conjecture. The system of parameters can be replaced by any system of parameters generating a smaller ideal than I. Therefore, in mixed characteristic the first parameter x_1 may be assumed to be p^t. (This step is not needed in characteritstic $p > 0$.) Call the system of parameters $x_1, \dots, x_d$. Now map the free resolution $G_\bullet$ of K to the flat resolution $\mathcal{K}_\bullet\big((x_1{}^\infty), \dots, (x_d{}^\infty); S\big)$ of the residue field L of S, lifting the induced map $K \to S$. Since $G_\bullet$ is free over R and $\mathcal{K}_\bullet\big((x_1{}^\infty), \dots, (x_d{}^\infty); S\big)$ is acyclic over S, this is possible. Composing this map of complexes with the map $\mathcal{K}_\bullet(x_1, \dots, x_d; R) \to G_\bullet$ that gives a supposed counterexample to the canonical element conjecture, we get a map

$$\mathcal{K}_\bullet(x_1, \dots, x_d; R) \to \mathcal{K}_\bullet\big((x_1{}^\infty), \dots, (x_d{}^\infty); S\big)$$

that lifts the obvious map $R/(x_1, \dots, x_d) \twoheadrightarrow K \hookrightarrow L$ and which is 0 in the d th spot.

On the other hand there is an obvious map

$$\mathcal{K}_\bullet(x_1, \dots, x_d; R) \to \mathcal{K}_\bullet\big((x_1{}^\infty), \dots, (x_d{}^\infty); S\big)$$

obtained by thinking of the Koszul complex as the tensor product of the d inclusion maps

$$0 \to x_j R \subseteq R \to 0,$$

and noting that each of these complexes is a subcomplex of

$$0 \to ({x_j}^{\infty}) \subseteq R \to 0.$$

Since the Koszul complex is R-free and the complex

$$\mathcal{K}_\bullet\big(({x_1}^{\infty}), \ldots, ({x_d}^{\infty}); S\big)$$

is acyclic, these two maps of complexes are homotopic. It follows that the image of the free generator $x_1 \cdots x_d$ of the dth term of the Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_d; R)$ maps to

$$(x_1, \ldots, x_d)\mathcal{K}_d\big(({x_1}^{\infty}), \ldots, ({x_d}^{\infty}); S\big),$$

which is $(x_1, \ldots, x_d)\big((x_1 \cdots x_d)^{\infty}\big)$.

Then for some $e \in \mathbb{N}$ we get an equation

$$x_1 \cdots x_d = (\sum_{j=1}^{d} x_j s_j)(x_1 \cdots x_d)^{1/p^e}.$$

Let $y_j = x_j^{1/p^e}$ for every j, and let $T \subseteq S$ be a module-finite extension of R that contains the y_j and the s_j, so that we have the equation

$$(y_1 \cdots y_d)^{p^e} = (\sum_{j=1}^{d} y_j^{p^e} s_j)(y_1 \cdots y_d),$$

in T. Dividing through by $y_1 \cdots y_d$ we have

$$(\#) \quad (y_1 \cdots y_d)^{p^e - 1} = \sum_{j=1}^{d} y_j^{p^e} s_j.$$

Let V be a coefficient ring for T, so that T is module-finite over $V[[y_2, \ldots, y_d]]$. Let $W = V[p^{1/p^e}]$. Then T is also module-finite over the regular local ring $A = W[[y_2, \ldots, y_d]$, and A contains all of the parameters $y_1, \ldots, y_d$. Since we are assuming the direct summand conjecture for A, we may apply an A-linear retraction to $(\#)$ to find that

$$(y_1 \cdots y_d)^{p^e - 1} = \sum_{j=1}^{d} y_j^{p^e} a_j$$

for elements $a_j \in A$. Since $y_1, \ldots, y_d$ is a regular sequence in A, this is a contradiction. $\square$