Math 711: Lecture of November 14, 2005

We can further reduce all of the equivalent conjectures in mixed characteristic to the special case of the direct summand conjecture where the ring consists of formal power series over a coefficient ring.

Theorem. *The following are equivalent for complete local domains* (R, m, K) *of mixed characteristic* $p > 0$ *and Krull dimension* d.

(1) *For every complete regular local ring* R *of the form* $V[[x_2, \ldots, x_d]]$, *where* V *is a coefficient ring (in mixed characteristic, a complete DVR of mixed characteristic* p *with maximal ideal* pV*),* R *is a direct summand of every module-finite extension ring.*

(2) *Whenever* R *is a complete local domain,* $H^d_m(R^+) \neq 0$.

(3) *For every complete regular local ring* R *of the form* $W[[x_2, \ldots, x_d]]$, *where* $W = V[p^{1/p^e}]$, $e \in \mathbb{N}$, *and* V *is a coefficient ring,* R *is a direct summand of every module-finite extension ring. In fact,* R *is a direct summand of* R^+.

Hence, all of these conditions are equivalent to the statement that for every local domain of mixed characteristic p *and Krull dimenson dimension* d, $\eta_R \neq 0$.

Proof. We have already seen that (3) is equivalent to the final statement, and so it will suffice to show that (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3). (3) $\Rightarrow$ (1) is clear, since we may take $e = 0$ in (3). To see that (1) $\Rightarrow$ (2), let V be a coefficient ring of R, and then R is module-finite over $A = V[[x_2, \ldots, x_d]$. Then $R^+ = A^+$, since R is module-finite over A. Since the maximal ideal m_A of A is such that $m_A R$ is primary to m,

$$H^d_m(R^+) \cong H^d_{m_A}(R^+) \cong H^d_{m_A}(A^+).$$

Since A is a direct summand over A of S for every module-finite extension S with $A \subseteq S \subseteq R^+$, we have that $H^d_{m_A}(A)$ injects into $H^d_{m_A}(S)$ for every such S. Taking the direct limit over all such S, we find that $H^d_{m_A}(A) \neq 0$ injects into

$$\varinjlim_S H^d_{m_A}(S) \cong H^d_{m_A}(\varinjlim_S S) \cong H^d_{m_A}(A^+),$$

which is consequently not 0.

It remains only to prove that (2) $\Rightarrow$ (3). In considering (3) it suffices to consider the case where the extension ring is an integral domain, and so it certainly suffices to show that R is a direct summand of R^+. Let $\pi = p^{1/p^e}$ denote the generator of the maximal ideal of W. Let $E = H^d_m(R)$, which is an injective hull for the residue class field of R. Then

$$H^d_m(R^+) \cong R^+ \otimes_R H^d_m(R) \cong R^+ \otimes_R E.$$

If this module is not zero, its dual into E is not zero. (Choose any nonzero cyclic submodule R/J. Then we have a nonzero surjection $R/J \twoheadrightarrow K$ and $K \hookrightarrow E$. The composition gives a nonzero map $R/J \to E$, which extends to the whole module because E is injective.) But

$$\mathrm{Hom}_R(R^+ \otimes)_R E,\, E) \cong \mathrm{Hom}_R\big(R^+,\, \mathrm{Hom}_R(E,\, E)\big) \cong \mathrm{Hom}_R(R^+,\, R)$$

using the adjointness of tensor and Hom and Matlis duality. Thus, there is a nonzero map $R^+ \to R$. Call the image of this map I, and choose h maximum such that $I \subseteq \pi^h R$, so that we can write $I = \pi^h I_0$, where I_0 is not contained in πR. Since $I \cong I_0$ as R-modules and we have a surjection $R^+ \twoheadrightarrow I$, we have a surjection $R^+ \twoheadrightarrow I_0$. Since every ideal of R is m-adically closed, we have that

$$\pi R = \bigcap_{N=1}^{\infty} (\pi,\, x_2^N,\, \dots\,,\, x_d^N)R,$$

and this shows that for some N, we have that I_0 is not contained in $(\pi,\, x_2^N,\, \dots\,,\, x_d^N)R$. This means that we have an R-linear map $\theta : R^+ \to R$ and $u \in R^+$ such that $\theta(u) \notin (\pi,\, x_2^N,\, \dots\,,\, x_d^N)R$. Consider the composition of θ with multiplication by u thought of as a map $R^+ \to R^+$. The composition $\phi : R^+ \to R$ is an R-linear map such that $\phi(1) \notin (\pi,\, x_2^N,\, \dots\,,\, x_d^N)R$.

Let $A_0 = W[[x_2,\, \dots\,,\, x_d^N]]$. The maximal ideal is $\mu = (\pi,\, x_2^N,\, \dots\,,\, x_d^N)A_0$, and expands to to $\mu R = (\pi,\, x_2^N,\, \dots\,,\, x_d^N)R$. Thus, $\phi(1) \in A - \mu A$. Since A has depth d over A_0, which is regular, A is free over A_0, and $\phi(1)$ is part of a free basis for A over A_0. Thus, there is an A_0-linear map $A^+ \to A_0$ that sends $\phi(1)$ to $1 \in A_0$. The composition with ϕ produces an A_0-linear map $\alpha : A^+ \to A_0$ that sends $1 \mapsto 1$, and so $A_0 \subseteq A^+$ splits. But since A is a module-finite extension of A_0, $A_0^+ = A^+$, and it follows that $A_0 \to A_0^+$ splits.

But there is a continuous W-isomorphism

$$A = W[[x_2,\, \dots\,,\, x_d]] \cong A_0 = W[[x_2^N,\, \dots\,,\, x_d^N]]$$

that sends $x_j \mapsto x_j^N$ for $2 \le j \le d$. This induces an isomorphism $A_0^+ \cong A^+$. Thus, the inclusion of $A_0 \hookrightarrow A_0^+$ is isomorphic with the inclusion of $A \hookrightarrow A^+$. Since the former splits, so does the latter. □

Theorem. *The monomial conjecture holds in every local ring of positive prime characteristic p. Hence, the canonical element conjecture holds for such rings as well.*

Proof. Suppose that $x_1, \dots, x_d$ is a system of parameters for $(R,\, m,\, K)$ such that

$$(x_1 \cdots x_d)^t \in (x_1^{t+1}, \dots, x_d^{t+1})R$$

for some $t \in \mathbb{N}$. Taking q th powers, where $q = p^e$, we find that

$$(x_1 \cdots x_d)^{qt} \in (x_1^{qt+q}, \dots, x_d^{qt+q})R$$

for all q. We can view $H_m^d(R) = \varinjlim_N R/(x_1^N,\, \dots, x_d^N)R$. In this direcct limit system, when we map

$$R/(x_1^q,\, \dots, x_d^q)R \to R/(x_1^{q+qt},\, \dots, x_d^{q+qt})R,$$

we multiply by $(x_1 \cdots x_d)^{qt}$, which is 0 in the image. Hence, the image of each of the modules $R/(x_1^q,\, \dots, x_d^q)R$ in $H_m^d(R)$ is 0, and since the terms $R/(x_1^q,\, \dots, x_d^q)R$ are cofinal in the direct limit system, we find that $H^d_m(R) = 0$. This is well-known to be false. In the interest of giving a relatively self-contained treatment, we deduce this fact from local duality. We may replace R by its completion and then it is module-finite over a regular local ring (A, m_A, K). Then $H^d_m(R) \cong H^d_{m_A}(R)$, and by local duality over A this is dual to $\mathrm{Hom}_A(R,\, A)$, which is nonzero even if we tensor with the fraction field of A. $\square$

We want to deduce the canonical element conjecture in equal characterisitc zero from the positive characteristic case. We cannot do this from the direct summand conjecture, because the needed implication only holds in residual characteristic $p > 0$, but we can use the following result, which is proved in the Lecture Notes for Math 711, Fall 2004: it is stated in the Lecture of December 3, and proved in the sequel, using Artin approximation. The version using only the condition in part (a) permits immediate reduction of many theorems for Noetherian rings containing a field to the case of local rings of affine algebras over a finite field. We assume this result here, and we shall use it to deduce the canonical element conjecture for all local rings R such that R_{red} contains a field from the characteristic $p > 0$ case, which we have already handled.

Theorem. *Consider a family of finite systems of polynomial equations over $\mathbb{Z}$ such that each system in the family involves variables $x_1, \ldots, x_d$ and other variables $Y_{i,1}, \ldots, Y_{i,h_i}$ where both h_i and the variables are allowed to depend on which system in the family one is considering. Suppose that none of these systems has either*

(a) *a solution in a finitely generated algebra over a finite field such that the values of the x_j generated an ideal of height d, nor*

(b) *a solution in a finitely generated algebra over a DVR of mixed characteristic $p > 0$ such that the ideal generated by the values of the x_j has height d.*

Then no system in the family has a solution in a Noetherian ring in such a way that the values of the x_j generate an ideal of height d. Moreover, (a) alone guarantees that there is no solution in a Noetherian ring containing a field such the values of the x_j generate an ideal of height d.

Note that once we have a solution in which the x_j generate an ideal of height d, we also have a solution in which the x_j are a system of parameters for a local ring of dimension d: simply localize at a height d minimal prime of the ideal generated by the x_j.

Theorem. *The canonical element conjecture holds for all local rings containing a field. More generally, it holds for any local ring R of Krull dimension d such that R has a quotient of Krull dimension d by a minimal prime P such that R/P contains a field.*

Proof. The final statement is immediate from the case where R contains a field. We may therefore assume that $(R,\, m,\, K)$ is such that R contains a field of characteristic 0. The existence of a counterexample implies that there is a system of parameters $x_1, \ldots, x_d$ for R together with a map from the Koszul complex $\mathcal{K}_\bullet = \mathcal{K}_\bullet(x_1, \ldots, x_d;\, R)$ to a free resolution $G_\bullet$ of K such that the induced map of augmentations is the canonical surjection $R/(x_1, \ldots, x_d) \twoheadrightarrow K$ and the map of free modules at the dth spot is 0. We may assume without loss of generality that $G_0 = R$ and that the map from $R = \mathcal{K}_0$ to $G_0 = R$ is the identity. We therefore have a commutative diiagram:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & G_d & \xrightarrow{\alpha_d} & G_{d-1} & \xrightarrow{\alpha_{d-1}} & \cdots & \xrightarrow{\alpha_2} & G_1 & \xrightarrow{\alpha_1} & R & \longrightarrow & 0 \\ & & {\scriptstyle 0}\Big\uparrow & & {\scriptstyle \phi_{d-1}}\Big\uparrow & & & & {\scriptstyle \phi_1}\Big\uparrow & & {\scriptstyle \mathrm{id}}\Big\uparrow & & \\ 0 & \longrightarrow & \mathcal{K}_d & \xrightarrow{\delta_d} & \mathcal{K}_{d-1} & \xrightarrow{\delta_{d-1}} & \cdots & \xrightarrow{\delta_2} & \mathcal{K}_1 & \xrightarrow{\delta_1} & R & \longrightarrow & 0 \end{array}$$

such that the image of G_1 in R is m. Note that we still have a counterexample to the canonical element conjecture if we weaken the condition on $G_\bullet$: it need only be a free complex, and we only need that the image of α_1 is contained in m, and then we have a set-up as in part (b) of the first Proposition from the Lecture Notes of November 9.

With this weakening in the conditions we can think of the counter-example to the canonical element conjecture coming from this map of complexes as arising from solving a system of polynomial equations over $\mathbb{Z}$. We fix the sizes of all the free modules. We let the entries of every α_i, $1 \le i \le d$ be unknown. We replace the x_i by corresponding labeled variables X_i in the matrices of the maps δ_i: each entry is either 0, or equal, up to sign, to one of the X_i. We let the entries of the matrices of the ϕ_i, $1 \le i \le d-1$, be unknowns as well. The condition that the upper row be a complex, i.e., that each $\phi_{i-1} \circ \phi_i = 0$, is given by polynomial equations over $\mathbb{Z}$. So is the condition that each square commute. Beyond that, each entry of the matrix of α_1 has a power that is a linear combination of the x_i. This condition can also be expressed using polynomials over $\mathbb{Z}$, introducing new variables to serve as the coefficients in the linear combinations.

It then follows that if these equations have a solution in a local ring of characteristic 0 such that the values of the X_i are a system of parameters, then the same is true in positive characteristic $p > 0$ for some p, contradicting the canonical element conjecture in characteristic $p > 0$. $\square$