Math 711: Lecture of November 16, 2005

Discussion: a second proof of the direct summand conjecture in characteristic $p > 0$. Before proceeding further, we give a different proof of the direct summand conjecture in positive prime characteristic p. As before, we may reduce to the complete case and so we may assume that the regular ring R has the form $K[[x_1, \ldots, x_d]]$, where K is a field and S is module-finite over K. Let L be a perfect field containing K, e.g., the perfect closure or algebraic closure of K. Then $R' = L[[x_1, \ldots, x_d]]$ is faithfully flat over R. Since $R \to S$ splits if and only if $\mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(R, R)$ is onto, an issue that is unaffected by applying $R' \otimes_R \_$, and since tensoring with a faithfully flat algebra commtes with Hom when the first module is finitely presented, we may consider whether $R' \to R' \otimes_R S$ splits instead, and so reduce to the case where K is perfect. When K is perfect we have that $R^q = K[[x_1^q, \ldots, x_d^q]]$ for every $q = p^e$ with $e \in \mathbb{N}$. As earlier, we may assume that S is a domain and, in particular, that S is torsion-free over R.

It follows that S embeds in $R^{\oplus h}$ for some h, and the image of $1 \in S$ will have some nonzero coordinate. Composing $S \hookrightarrow R^h$ with the projection of $R^{\oplus h}$ on R using that coordinate, we have an R-linear map $\phi : S \to R$ such that $\phi(1) \neq 0$, and so we can choose $q = p^e$ such that $\phi(1) \notin (x_1^q, \ldots, x_d^q)R$. R is free over R^q: the monomials in the x_j with every exponent less than q give a free basis. When we expand the maximal ideal m_q of R^q to R, we get $(x_1^q, \ldots, x_d^q)R$, and so $\phi(1) \notin m_q R$ is part of a free basis for R over R^q. We may therefore choose an R^q-linear map $\beta : R \to R^q$ that sends $\phi(1) \mapsto 1$. Then $\beta \circ \phi : S \to R^q$ is R^q-linear and sends $1 \mapsto 1$. We may evidently restrict this map to S^q, and so obtain an R^q-linear map $S^q \to R^q$ that splits $R^q \hookrightarrow S^q$. But we have a commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\cong} & S^q \\ \uparrow & & \uparrow \\ R & \xrightarrow{\cong} & R^q \end{array}$$

where the horizontal isomorphisms are obtained by restricting the iterated Frobenius endomorphism $F^e : u \mapsto u^q$ and the vertical maps are inclusions. Since the map $R^q \hookrightarrow S^q$ is split over R^q, the isomorphic map $R \hookrightarrow S$ is split over R. $\square$

In any case, we have now proved the direct summand, monomial, and canonical element conjectures in equal characteristic. We next note the following fact:

Theorem. *If the local ring R has a big Cohen-Macaulay module M, then $\eta_R \neq 0$.*

Proof. What we need about M is that some system of parameters $x_1, \ldots, x_d$ for R is a regular sequence on M. This hypothesis includes the condition that $(x_1, \ldots, x_d)M \neq M$. If $\eta_R = 0$ then for some t we have a map ϕ_d of complexes $\mathcal{K}_\bullet = \mathcal{K}(x_1^t, \ldots, x_d^t; R) \to G_\bullet$, where $G_\bullet$ is a free resolution of K such that $G_0 = R$, lifting $R/(x_1^t, \ldots, x_d^t)R \twoheadrightarrow K$, such that $\phi_d = 0$. Choose $v \in M$ such that $v \notin (x_1^t, \ldots, x_d^t)M$, but m kills the image u of v in $M/(x_1^t, \ldots, x_d^t)M$. This is possible because, while $M/(x_1^t, \ldots, x_d^t)M$ is nonzero, every nonzero element is killed by a power of m, and so each nonzero cyclic submodule has a non-trivial socle. We therefore have a map $K \to M/(x_1^t, \dots, x_d^t)M$ that takes $1 \in K$ to u, and we can lift this to the map $G_0 = R \to M$ that sends $1 \mapsto v$. Since $G_\bullet$ is free and $\mathcal{K}'_\bullet = \mathcal{K}_\bullet(x_1^t, \dots, x_d^t;\, M)$ is acyclic (this follows because $x_1^t, \dots, x_d^t$ is a regular sequence on M), the map $R \to M$ sending $1 \mapsto v$ extends to a map of complexes $G_\bullet \to \mathcal{K}'_\bullet$. The compostion of $\mathcal{K}_\bullet \to G_\bullet$ with this map $G_\bullet \to \mathcal{K}'_\bullet$ gives a map $\theta_\bullet : \mathcal{K}_\bullet \to \mathcal{K}'_\bullet$ such that $\theta_0 : R \to M$ sends $1 \mapsto v$ while $\theta_d = 0$. We obtain another map $\zeta : \mathcal{K}_\bullet \to \mathcal{K}'_\bullet$ simply by tensoring the map $\theta_0 : R \to M$ with the complex $\mathcal{K}_\bullet$. The maps $\theta_\bullet$ and $\zeta_\bullet$ agree in degree 0. Since $\mathcal{K}_\bullet$ is free and $\mathcal{K}'_\bullet$ is acyclic, $\theta_\bullet$ and $\zeta_\bullet$ are homotopic, and so there exists a map $h : \mathcal{K}_{d-1} \to \mathcal{K}'_d$ such that $\zeta_d - \theta_d = h \circ \delta_{d-1}$, where $\delta_{d-1} : \mathcal{K}_d \to \mathcal{K}_{d-1}$ has a matrix in which each entry is $\pm x_j^t$ for some j. This implies that $\zeta_d(1) - \theta_d(1) = v - 0 = v$ is in $(x_1^t, \dots, x_d^t)M$, a contradiction, since u is nonzero in $M/(x_1^t, \dots, x_d^t)M$. $\square$

From this result, we can conclude that the canonical element conjecture holds in dimension at most two as well as in equal characteristic, since it reduces to the case of normal complete local domains, which are Cohen-Macaulay in dimension two. By a difficult result due to Ray Heitmann, the direct summand conjecture holds in dimension three in mixed characeristic, so that the canonical element conjecture is also true in dimension three.

We next want to show that the canonical element conjecture implies the improved new intersection conjecture.

Theorem. *Let (R, m, K) be a local ring of Krull dimension d such that $\eta_R \neq 0$. Then the improved new intersection theorem holds for R: that is, if $G_\bullet$ is a finite complex of finitely generated free R-modules of length n, say*

$$0 \to G_n \to \cdots \to G_0 \to 0$$

such that $H_0(G_\bullet) \neq 0$ has a minimal generator u that is killed by a power of m and $H_i(G_\bullet)$ has finite length for $i \geq 1$, then $\dim(R) = d \leq n$.

Proof. Let v be a minimal generator of G_0 such that v maps to $u \in M = H_0(G_\bullet)$. Since u is killed by a power of m, there exists a system of parameters $x_1, \dots, x_d$ for R such that there is a surjection $R/(x_1, \dots, x_d) \to Ru$: this lifts to a map $R \to G_0$ such that $1 \mapsto v$. Let $\mathcal{K}_\bullet^{(t)}$ denote $\mathcal{K}_\bullet(x_1^t, \dots, x_d^t; R)$ for each $t \geq 1$. Then we have the beginning of a map of complexess:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_n & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & 0 \\
 & & & & & & {\scriptstyle 1\mapsto v}\uparrow & & \\
\mathcal{K}_{n+1}^{(1)} & \longrightarrow & \mathcal{K}_n^{(1)} & \longrightarrow & \cdots & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

such that the vertical map $R \to G_0$ induces the specified map of the augmentations. It we were able to extend this to a map of complexes (we would be able to do so if the upper row were acyclic, for example) we would have a contradiction when $d > n$, for the resulting map of complexes violates the canonical conjecture in the form given in part (b) of the first Proposition in the Lecture Notes of November 9. The point is that since $d > n$, the map at the d th spot is certainly 0. We cannot quite do this: instead, we shall show that one can give such a map of complexes when the bottom row is replaced by $\mathcal{K}_\bullet^{(t)}$ for any sufficiently large value of t, which is still enough to violate the canonical element conjecture.

We shall show by induction that we can carry out the construction of the map of complexes through the i th spot for all $i \geq 1$. At every step, we shall allow t to increase in constructing the next map. Therefore, we may assume for some $i \geq 1$ that we have constructed a map

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & G_n & \longrightarrow & \cdots & \longrightarrow & G_i & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & 0 \\
 & & & & & & {\scriptstyle\phi_i}\Big\uparrow & & & & {\scriptstyle\phi_0}\Big\uparrow & & \\
\mathcal{K}_{n+1}^{(t)} & \longrightarrow & \mathcal{K}_n^{(t)} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}_i^{(t)} & \longrightarrow & \cdots & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

where ϕ_0 is the map $1 \mapsto v$. For each $h \geq 1$ we have a standard map $\theta_\bullet : \mathcal{K}_\bullet^{(t+h)} \to \mathcal{K}_\bullet^{(t)}$ which is the identity in degree 0, and in degree 1 is given by the diagonal $d \times d$ matrix whose diagonal entries are the elements $x_1^h, \ldots, x_d^h$. In higher degree it is given by the exterior powers of the matrix that gives the map in degree 1. The only thing we need to know is that for every $j \geq 1$, $\mathrm{Im}\,(\theta_j) \subseteq m^h\mathcal{K}_j^{(t)}$. Let $Z_i = \mathrm{Ker}\,(G_i \to G_{i-1})$ and $B_i = \mathrm{Im}\,(G_{i+1} \to G_i)$. Choose N such that $m^N H_i(G_\bullet) = 0$. Then $m^N Z_i \subseteq B_i$. By the Artin-Rees Lemma, we can choose $c \in \mathbb{N}$ such that for all $s \geq c$, $m^s G_i \cap Z_i \subseteq m^{s-c} Z_i$. Let $h = N + c$. Then $m^h Z_i \subseteq m^N Z_i \subseteq B_i$.

We have a map $\mathcal{K}_\bullet(t) \to G_\bullet$ defined through the i th spot. We get a composite map $\mathcal{K}_\bullet^{(t+h)} \to \mathcal{K}_\bullet^{(t)} \to G_\bullet$ defined through the i th spot as well. We claim that this map can be extended to one defined at the $i+1$ st spot. We have:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & G_{i+1} & \longrightarrow & G_i & \longrightarrow & G_{i-1} & \longrightarrow & \cdots \\
 & & & & {\scriptstyle\phi_i}\Big\uparrow & & {\scriptstyle\phi_{i-1}}\Big\uparrow & & \\
 & & & & \mathcal{K}_i^{(t)} & \longrightarrow & \mathcal{K}_{i-1}^{(t)} & \longrightarrow & \cdots \\
 & & & & {\scriptstyle\theta_i}\Big\uparrow & & {\scriptstyle\theta_{i-1}}\Big\uparrow & & \\
\cdots & \longrightarrow & \mathcal{K}_{i+1}^{(t+h)} & \xrightarrow{\delta_{i+1}} & \mathcal{K}_i^{(t+h)} & \xrightarrow{\delta_i} & \mathcal{K}_{i-1}^{(t+h)} & \longrightarrow & \cdots
\end{array}$$

To fill in the required map $\mathcal{K}_{i+1}^{(h+t)} \to G_{i+1}$, it suffices to show that for each generator b in a free basis for $\mathcal{K}_{i+1}^{(t+h)}$, $\phi_i\theta_i\delta_{i+1}(b)$ is in B_i: we can then choose $g \in G_{i+1}$ that maps to it, and g will serve as the image of b under the map $\mathcal{K}_{i+1}^{(t+h)} \to G_{i+1}$ that we are trying to construct. Note that this element is a cycle: its image in G_{i-1} can also be computed by traversing two other edges of the rightmost rectangle, and $\delta_{i+1}\delta_i(b) = 0$. Because $\mathrm{Im}\,(\theta_i) \subseteq m^h\mathcal{K}_i^{(t)}$, we have that $\mathrm{Im}\,(\phi_i\theta_i) \subseteq m^h G_i$, and so $\phi_i\theta_i\delta_{i+1}(b) \in m^h G_i \cap Z_i \subseteq B_i$, as required. This completes the proof of the inductive step, and the result follows. $\square$

We have already seen that the new intersection conjecture (when $H_0(G_\bullet) \neq 0$ has finite length) and the intersection theorem (if M, N are finitely generated over R local and

$\ell(M \otimes_R N)$ is finite, then $\dim(N) \leq \mathrm{pd}_R M$, which we might as wll assume is finite) follow from the improved new intersection theorem. (For the latter, with $I = \mathrm{Ann}_R N$, one may apply the new intersection theorem to the (R/J)-free complex obtained by applying $R/J \otimes_R \_$ to a minimal R-free resolution of M.) Note that the dimension theorem controls the dimension of the support of N when $\mathrm{Supp}\,(N)$ and $\mathrm{Supp}\,(M)$ intersect only at the closed point of R.

We want to show that the zerodivisor conjecture follows from the intersection theorem. We need one preliminary:

Lemma. *Let N be an a finitely generated R-module, where (R, m, K) is local. Then for every nonzero submodule $W \subseteq N$, $\dim(W) \geq \mathrm{depth}_m N$.*

Proof. We use induction on $d = \mathrm{depth}_m N$. If it is 0, the result is obvious. Assume $d \geq 1$, and suppose there is a submodule W_0 whose dimension is $< d$. Let $x \in m$ be a nonzerodivisor on N, and consider the increasing chain $W_0 :_N x^t$. This will be stable for large t. Call the stable value W. Note that since, for $t \gg 0$, $W \cong x^t W \subseteq W_0$, we have that $\dim(W) < d$. Observe that x is not a zerodivisor on N/W, and it was chosen so as not to be a zerodivisor on N, and, hence, not on W. Then W/xW injects into N/xN: since the latter has depth $d-1$, we have that $\dim(W) - 1 = \dim(W/xW) \geq \mathrm{depth}_m(N/xN)$ (by the induction hypothesis) $= d-1$, and so $\dim(W) \geq \mathrm{depth}_m N$, a contradiction. $\square$

We can now prove:

Theorem (Peskine-Szpiro). *If the intersection theorem holds for all localizations of (R, m, K), and $M \neq 0$ is a finitely generated module of finite projective dimension, then every zerodivisor in R is a zerodivisor on M.*

Proof. It is equivalent to prove that every associated prime P of R is contained in an associated prime of M. Assume that one has a counterexample, and localize at a minimal prime of $P + \mathrm{Ann}_R M$. The new local ring is still a counterexample. Therefore, we may assume that $P + \mathrm{Ann}_R M$ is primary to m, which means that $R/P \otimes_R M$ has finite length. Then, by the intersection theorem, $\dim(R/P) \leq \mathrm{pd}_R M$. By the Lemma just above, $\mathrm{depth}_m R \leq \dim(R/P)$. But then $\mathrm{depth}_m R \leq \mathrm{pd}_R M$. But we always have the other inequality, since $\mathrm{depth}_m M + \mathrm{pd}_R M = \mathrm{depth}_m R$, and so we have equality, and $\mathrm{depth}_m M = 0$. But then m is an associated prime of M and contains P, a contradiction. $\square$