Math 711: Lecture of November 18, 2005

Bass raised the following question: if a local ring has a nonzero finitely generated module of finite injective dimension, must the ring be Cohen-Macaulay. We are aiming to prove next a result of Peskine and Szpiro, which is that the intersection theorem implies an affirmative answer to Bass's question. It is the case, as we shall see below, that a Cohen-Macaulay local ring always has such a module.

We first summarize some basic facts about injective modules over a Noetherian ring.

An extension of modules $M \subseteq Q$ is called *essential* if, equivalently:

(1) Every nonzero submodule of Q has nonzero intersection with M.
(2) Every nonzero element of Q has a nonzero multiple in M.
(3) Every R-linear map defined on Q that is injective when restricted to M is injective on Q.

We note that if $M \hookrightarrow Q$ is essential, $\mathrm{Ass}\,(Q) = \mathrm{Ass}\,(M)$, for if $u \in Q$ is such that $Ru \cong R/P$, the nonzero multiple of u in M also has annihilator P.

By the the final Theorem in the Lecture Notes from March 19, Math 615, Fall 2004, every R-module M can be embedded in an injective module E. Within E, by Zorn's lemma, there is a maximal essential extension E_0 of M. But E_0 is then a maximal essential extension of M in an absolute sense, for if $E_0 \to W$ were a proper essential extension, the map $E_0 \to E$ would extend to W and be injective on W by (3), so that E_0 would have a proper essential extension within E.

A submodule E_1 of E maximal with respect to being disjoint from E_0 must be a complement of E_0, i.e., $E_0 \oplus E_1 = E$: if the map $E_0 \to E/E_1$ were not an isomorphism, it would be a proper essential extension of E_0, by the maximality of E_1. Thus, we have shown that E_0 is injective, and so we have prove that an essential extension of M that is injective always exists, and is a direct summand of any injective module containing M. E_0d is unique up to non-unique isomorphism, and is called an *injective hull* of M over R. We write $E_R(M)$ or simply $E(M)$ for an injective hull of M over R.

We say that an acyclic complex

$$0 \to E^0 \to E^1 \to \cdots \to E^i \to \cdots$$

is a *minimal injective resolution* of M if the E_i are all injective, $i \geq 0$,

$$0 \to M \to E^0 \to E^1$$

is exact, $M \hookrightarrow E_0$ is an injective hull of M, and, for all $i \geq 1$, E_i is an injective hull of the image of E_{i-1}. Minimal injecctive resolutions always exist, and are unique up to nonunique isomorphism.

We now consider the case where the ring is Noetherian. Note that an essential extension $M \hookrightarrow Q$ remains essential after localization at a multiplicative system W: if $u/w \in W^{-1}Q$

is nonzero, where $u \in Q$ and $w \in W$, then u has a nonzero multiple v in R such that the annihilator of $v/1 \in W^{-1}Q$ is a prime ideal P of R disjoint from W. Then $Ru \cap M \neq 0$, and if $y \in Ru \cap M - \{0\}$, its annihilator in R is P, and so $y/1 \in W^{-1}Q$ is nonzero.

Over a Noetherian ring, a direct sum of injective modules is injective, and every injective module is a direct sum of modules of the form $E(R/P)$, where P is prime. When P is prime, $E_R(R/P)$ has, in a unique way, the structure of an R_P module, and is also $E_{R_P}(R_P/PR_P)$. P is the only associated prime of $E(R/P)$, since the assassinator of $E(R/P)$ is the same as $\mathrm{Ass}\,(R/P)$. Moreover, every element of $E(R/P)$ is killed by a power of P. Therefore, this module is also a module over the (PR_P)-adic completion $(S,\, m)$ of R_P, and it is also $E_S(S/m)$.

For every pair of prime ideals P and Q of R either $P \subseteq Q$ and $E(R/P)_Q \cong E(R/P)$ or there is an element of P not in Q, in which case $E(R/P)_Q = 0$. It follows that the localization of an injective R-module at a prime Q is again injective, both over R and over R_Q. It also follows that when one localizes a minimal injective resolution at a prime P, it remains a minimal injective resolution over R_P.

Thus, given any module M over a Noetherian ring R, it has a minimal injective resolution $0 \to E^0 \to E^1 \to \cdots E^i \to \cdots$ and, for each i, E^i has the form

$$\bigoplus_{P \in \mathrm{Spec}(R)} E(R/P)^{\oplus \mu_i(P,M)}$$

for suitable cardinal numbers $\mu_i(P,M)$, which may be infinite. We want to see that these are independent of any choices made, and that if M is finitely generated the numbers $\mu_i(P,\, M)$ are finite. Since one obtains a minimal injective resolution of M_P over R_P when one localizes at P, it suffices to show that the $\mu_i(P,M)$ are well-defined when $(R,\, m,\, K)$ is local and $P = m$.

Let $(R,\, m,\, K)$ be local. Then the socle in any module M is the same as the socle in $E(M)$, since any nonzero element of $E(M)$ has a nozero multiple in M. It follows that in an injective resolution of M, every element of the socle in any injective maps to 0 in the next injective. Therefore, when we compute $\mathrm{Ext}^\bullet_R(K,\, M)$ by applying $\mathrm{Hom}_R(K,\, \_)$ to a minimal injective resolution of M, we get a complex of K-vector spaces in which the maps are all 0. Note that

$$\mathrm{Hom}_R(W,\, \bigoplus_{\lambda \in \Lambda} V_\lambda) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_R(W,\, V_\lambda)$$

even when Λ is infinite provided that W is finitely generated. Hence,

$$\mathrm{Hom}_R\big(K,\, \bigoplus_{P \in \mathrm{Spec}(R)} E(R/P)^{\oplus \mu_i(P,M)}\big) \cong \bigoplus_{P \in \mathrm{Spec}(R)} \mathrm{Hom}_R\big(K,\, E(R/P)\big)^{\oplus \mu_i(P,M)}.$$

If $P \neq m$, then then no nonzero element of $E(R/P)$ is killed by P, and therefore we have that $\mathrm{Hom}_R\big(K, E(R/P)\big) = 0$. If $P = m$, $\mathrm{Hom}_R\big(K,\, E(R/m)\big) \cong K$. It follows that $\mathrm{Hom}_R(K,\, E^i) \cong K^{\oplus \mu_i(m,\, M)}$. As already mentioned, this is a complex of K-vector spaces in which all the maps are 0. It follows from the definition that $\mathrm{Ext}^i_R(K,\,M) \cong K^{\oplus \mu_i(m,M)}$, which gives the uniqueness result we want for the numbers $\mu_i(m,\,M)$.

The number $\mu_i(P,\,M)$ of copies of $E(R/P)$ occurring in the direct sum that gives E^i, the i th term of a minimal injective resolution of M, is likewise unique, by the localization argument mentioned above. With $\kappa = \kappa_P = R_P/PR_P$, we may recover this number as $\dim_\kappa \mathrm{Ext}^i_{R_P}(\kappa,\,M_P)$: we are applying the result already obtained for the maximal ideal to M_P over the ring R_P. When M is finitely generated, it follows that $\mu_i(P,\,M)$ is finite, and is referred to as the i th *Bass number* $\mu_i(P,\,M)$ of M with respect to P. If R is local and P is omitted from the notation, P is understood to be the maximal ideal m of R.

We now want to discuss briefly the situation for Cohen-Macaulay rings. We first consider the case where $(R,\,m,\,K)$, of dimension d, is a homomorphic image of a Gorenstein local ring S of dimension n. In this case, one can define a so-called *canonical module* ω as $\mathrm{Ext}^{n-d}_S(R,\,S)$. One can show that $H^d_m(\omega) \cong E_R(K) = E$ and that $\mathrm{Hom}_R(\omega,\,E) \cong H^d_m(R)$. Thus, if $\underline{x} = x_1,\,\ldots\,,x_d$ is a system of parameters for R, $\mathcal{C}_\bullet(\underline{x}^\infty; R)$, numbered backwards, is a flat resolution of $H^d_m(R) = \mathrm{Hom}_R(\omega,\,E)$, and so $H^j_m(M) \cong \mathrm{Tor}^R_{d-j}\big(M,\,H^d_m(R)\big)$. If M is finitely generated, this is the same as $\mathrm{Ext}^{d-j}_R(M,\,\omega)^\vee$, where $\_^\vee$ denotes the functor $\mathrm{Hom}_R(\_\,,\,E)$, by exactly the same argument as in our earlier proof of local duality (the Theorem at the bottom of the third page of the Lecture Notes from November 4). We state this explicitly:

Theorem (local duality for Cohen-Macaulay rings). *If $(R,\,m,\,K)$ is a Cohen-Macaulay local ring with canonical module ω as described in the paragraph above, and $\_^\vee$ is $\mathrm{Hom}_R\big(\_\,,\,E_R(K)\big)$, then for every finitely generated module M, $H^j_m(M) \cong \mathrm{Ext}^{d-j}(M,\,\omega)^\vee$ for all integers j.*

We recall from the third Proposition on page 5 of the Lecture Notes from March 22, Math 615, Winter 2004, that $\mathrm{id}_R N \leq k$ if and only if $\mathrm{Ext}^j_R(R/I,\,N) = 0$ for every ideal I of R and for all $j \geq k+1$. The fact that $\mathrm{id}_R N \leq k$ is also characterized by the vanishing of $\mathrm{Ext}^j_R(M,N)$ for all R-modules M and all $j \geq k+1$, or for all finitely generated R-modules M. In the Noetherian case, since every finitely generated R-module has a prime cyclic filtration, and since one has the long exact sequence for Ext, it suffices to impose the same condition when I is a prime ideal of R. Note that we have, in consequence:

Corollary. *If R is a Cohen-Macaulay local ring of Krull dmension d with canonical module ω, then ω is a finitely generated R-module of finite injective dimension.*

Proof. For $j > d$, if M is finitely generated, $\mathrm{Ext}^j_R(M,\omega)$ has dual $H^{d-j}_m(M) = 0$, and so $\mathrm{Ext}^j_R(M,\,\omega) = 0$ for every finitely generated R-module M if $j > d$. $\square$

Now, if M has finite projective dimension over the Cohen-Macaulay ring R with canonical module ω then, simply because ω has depth $d = \dim(R)$, $\mathrm{Tor}^R_i(M,\omega) = 0$ for $i \geq 1$. Therefore, if M is a finitely generated R-module of finite projective dimension, one has that $M \otimes_R \omega$ has a finite resolution by direct sums of copies of ω, obtained by tensoring the finite free resolution of M with ω over R. It follows that each such module $M \otimes_R \omega$ is finitely genereated of finite injective dimension. In fact, it turns out that over the Cohen-Macaulay ring R, if there is a canonical module ω, the functors $M \mapsto M \otimes_R \omega$ and

$N \mapsto \mathrm{Hom}_R(\omega,\, N)$ give a covariant equivalence between the category of finitely generated modules of finite projective dimension over R and the category of finitely generated modules of finite injective dimension over R. This gives a reasonably complete understanding of what the finitely generated modules of finite injective dimension are. Even when there is not necessarily a canonical module, $E_{R/(\underline{x})R}(K)$ is a finite length module of finite injective dimension over R. In fact, if ω denotes a canonical module for $\widehat{R}$, this is the same as $\omega/(\underline{x})\omega$, which evidently has finite injective dimension over $\widehat{R}$. We shall see in part (f) of the Theorem below that this implies its injective dimension over R is finite.

It turns out to be quite difficult, however, to show that when a local ring R has a finitely generated nonzero module of finite injective dimension, then R is Cohen-Macaulay. We begin the needed analysis now. Our first goal is to show that whenever a local ring $(R,\, m,\, K)$ possesses a finitely generated module $N \neq 0$ with $\mathrm{id}_R N < \infty$, we have $\mathrm{id}_R N = \mathrm{depth}_m R$. The proof needs several lemmas.

Lemma. *Let $(R,\, m,\, K)$ be local and suppose that M is finitely generated with $\mathrm{pd}_R M = 1$. If N is finitely generated and $\mathrm{Ext}^1_R(M,\, N) = 0$, then $N = 0$.*

Proof. Let $0 \to R^b \xrightarrow{A} R^a \to M \to 0$ be a minimal free resolution of M: then A has entries in M. But $\mathrm{Ext}^1_R(M,\, N)$ is the cokernel of $N^a \xrightarrow{A^*} N^b$, where A^* is the transpose of the matrix A. The image of the map is contained in mN^b. By Nakayama's lemma, if the cokernel vanishes then $N^b = 0$ and so $N = 0$. $\square$

Lemma. *Let $(R,\, m,\, K)$ be local with $\mathrm{depth}_m R = d$, and let $N \neq 0$ be a finitely generated R-module with $\mathrm{id}_R N < \infty$. Then $\mathrm{id}_R N \geq d$.*

Proof. Assume that $\mathrm{id}_R N < d$. Let $M = \mathrm{syz}^{d-1} R/(\underline{x})R$, where $\underline{x} = (x_1,\, \ldots\, , x_d)$ is a maximal regular sequence in M. Then $\mathrm{pd}_R M = 1$, and $\mathrm{Ext}^1(M,\, N) \cong \mathrm{Ext}^d(R/(\underline{x})R,\, N) = 0$, since $\mathrm{id}_R N < d$, contradicting the preceding Lemma. $\square$

We shall prove the other inequality a bit later. We next note:

Theorem. *Let $(R,\, m,\, K)$ be local and let N be a finitely generated R-module.*

(a) *If $\mathrm{Ext}^j_R(R/Q,\, N) = 0$ for all $j \geq s+1$ (respectively, for $j = s+1$), and for all primes Q strictly containing a prime P, then $\mathrm{Ext}^j_R(R/P,\, N) = 0$ for all $j \geq s$ (respectively, for $j = s$).*
(b) *In particular, if $\mathrm{Ext}^{s+1}_R(K, N) = 0$, then for every prime P such that $\dim(R/P) = 1$, $\mathrm{Ext}^s_R(R/P,\, N) = 0$.*
(c) *If $\mathrm{Ext}^j_R(K, N) = 0$ for all $j \geq s+1$, where $s \in \mathbb{N}$, then $\mathrm{id}_R(N) \leq s$.*
(d) *If $P \subset Q$ are distinct primes of R with no prime strictly between them, and $\mu_k(P, N) \neq 0$, then $\mu_{k+1}(Q, N) \neq 0$.*
(e) *If $P \subseteq Q$ are primes with $\dim(R_Q/PR_Q) = h$, then $\mu_k(P, M) \neq 0$ implies that $\mu_{k+h}(Q, M) \neq 0$.*
(f) *The module N has finite injective dimension over R if and only if $\widehat{N}$ has finite injective dimension over $\widehat{R}$, and the injective dimensions are the same.*

Proof. (a) Since there are primes strictly containing P, we con choose $x \in m - P$, and then there is a short exact sequence

$$(*) \quad 0 \to R/P \xrightarrow{x} R/P \to R/(P + xR) \to 0.$$

Then $M = R/(P + xR)$ has a prime cyclic filtration by primes strictly containing P. It follows that $\mathrm{Ext}^{j+1}(M, N) = 0$, from the long exact sequence for Ext. But then the short exact sequence displayed yields a long exact sequence part of which is:

$$\cdots \to \mathrm{Ext}^j_R(R/P,\, N) \xrightarrow{x} \mathrm{Ext}^j_R(R/P,\, N) \to \mathrm{Ext}^{j+1}_R(M, N) \to \cdots$$

and, since the third term is 0, we obtain that $\mathrm{Ext}^j_R(R/P,\, N) = x\mathrm{Ext}^j_R(R/P,\, N)$. By Nakayama's lemma, we have that $\mathrm{Ext}^j_R(R/P,\, N) = 0$. (b) is a special case of (a), since m is the only prime of R strictly containing P.

(c) follows from (a) because we may show by induction on i that $\mathrm{Ext}^j(R/P, N) = 0$ for all P such that R/P has height i and all $j \geq s + 1$.

For part (d), we may replace R and N by R_Q and N_Q, and so assume that $Q = m$. If $\mu_{k+1}(m, N) = 0$, then $\mathrm{Ext}^{k+1}_R(K, N) = 0$, and we obtain from the parenthetical form of part (a) that $\mathrm{Ext}^k_R(R/P,\, N) = 0$. This remains true when we localize at P, which shows that $\mu_k(R/P, N) = 0$, a contradiction.

Part (e) is immediate from part (d): there is a saturated chain of length h, say $P = Q_0 \subset Q_1 \subset \cdots \subset Q_h = Q$, joining P to Q, where the inclusions are strict and there is no prime strictly between Q_i and Q_{i+1} for each i, and the result follows by induction on h.

Part (f) follows from part (c), since the modules $\mathrm{Ext}^j_R R(K, N)$, which are finite-dimensional K-vector spaces, are essentially unchanged by completion. □

We can now prove:

Theorem. *Let (R, m, K) be local and let $N \neq 0$ be a finitely generated module with $\mathrm{id}_R N < \infty$. Then $\mathrm{id}_R N = \mathrm{depth}_m R$.*

Proof. We have already shown that $\mathrm{id}_R N \geq d = \mathrm{depth}_m R$. To get the opposite inequality, suppose that $\mathrm{id}_R N = s > d$. Then $\mathrm{Ext}^j(K, N) = 0$ for $j > s$ while $\mathrm{Ext}^s(K, N) \neq 0$. Let $\underline{x} = x_1, \ldots, x_d$ be a maximal regular sequence in R. Then $R/(\underline{x})$ has depth 0, and so we have a short exact sequence $0 \to K \to R/(\underline{x}) \to C \to 0$ for some R-module C. The long exact sequence for Ext then yields

$$\cdots \to \mathrm{Ext}^s_R(R/(\underline{x}),\, N) \to \mathrm{Ext}^s_R(K,\, N) \to \mathrm{Ext}^{s+1}(C,\, N) \to \cdots.$$

The term on the left vanishes because $\mathrm{pd}_R R/(\underline{x}) = d < s$, while the term on the right vanishes because $s + 1 > \mathrm{id}_R N$. It follows that $\mathrm{Ext}^s_R(K,\, N) = 0$, a contradiction. □