Math 711: Lecture of November 21, 2005

We continue with our program of showing that an affirmative answer to Bass's question follows from the intersection theorem of Peskine and Szpiro.

Lemma. *If $(R,\, m,\, K)$ is a local ring and M a finitely generated module with $\mathrm{Ann}_R M = I$, then*

$$\mathrm{depth}_m R \leq \mathrm{depth}_I R + \dim(M) \leq \dim(R).$$

Proof. For the right hand inequality, choose P in the support of M such that $\dim(R/P) = \dim(M)$. Then $\mathrm{depth}_I R + \dim(M) \leq \dim(R_P) + \dim(M) = \mathrm{height}\,(P) + \dim(R/P) \leq \dim(R)$.

For the other inequality, one may kill a maximal regular sequence in I: both sides decrease by the same amount, since M is unchanged. Therefore we may assume that I has depth 0, and is contained in an associated prime P of 0 in R. Then $\dim(M) = \dim(R/P) \geq \mathrm{depth}_m R$. $\square$

Lemma. *If N is a finitely generated nonzero R-module such that $\mathrm{id}_R N < \infty$, then for every prime P in the support of N, $\dim(R/P) + \mathrm{depth}(R_P) = \mathrm{depth}_m R$.*

Proof. We have that $\mathrm{id}_{R_P} N_P = \mathrm{depth}_{PR_P}(R_P) = k$, say. Then, with $\kappa_P = R_P/PR_P$, we have that $\mathrm{Ext}^k_{R_P}(\kappa_P, N_P) \neq 0$, and so $\mu_{k+h}(K,\, N) \neq 0$, where $h = \dim(R/P)$. But then $k + h \leq \mathrm{id}_R N = \mathrm{depth}_m R$. But, by the preceding Lemma,

$$\mathrm{depth}_m R \leq \mathrm{depth}_P R + \dim(R/P) \leq \mathrm{depth}(R_P) + \dim(R/P).$$

$\square$

Discussion: the dual into the injective hull is independent of the choice of ring. We want to establish the statement of the heading for module-finite local extensions. One of the main points is that when $(R,\, m,\, K) \to (S,\, n,\, L)$ is local map of local rings such that S is module-finite over the image of R, then $E_S(L) \cong \mathrm{Hom}_R\big(S,\, E_R(K)\big)$. The latter is injective over S by the Corollary that is the next to last result in the Lecture Notes from March 19, Math 615, Fall 2004. We restate that result, although we have changed the names of the rings to suit the current situation:

Corollary. *Let S be an R-algebra, let F be a flat S-module, and let E be an injective R-module. Then $\mathrm{Hom}_R(F,\, E)$ is an injective S-module.*

We may apply this with $E = E_R(K)$ and $F = S$. We get that $E' = \mathrm{Hom}_R\big(S,\, E_R(K)\big)$ is injective over S, and by the adjointness of tensor and Hom we also get that $\mathrm{Hom}_S(\_\,,\, E')$ and $\mathrm{Hom}_R(\_\,,\, E)$ are the same functor on S-modules (for a given S-module M,

$$\mathrm{Hom}_S(M,\, E') = \mathrm{Hom}_S\big(M,\, \mathrm{Hom}_R(S,\, E)\big) \cong \mathrm{Hom}_R(M \otimes_S S,\, E) \cong \mathrm{Hom}_R(M,\, E),$$

as required.) Note that every element of E' is killed by a power of m, and, hence, by a power of n. It follows that E' is a direct sum of copies of $E_S(L)$. But $\mathrm{Hom}_S(L, E') \cong \mathrm{Hom}_R(L, E) \cong \mathrm{Hom}_K(L,\, K)$: as a K-vector space, this has the same dimension as L, and so as an L-vector space, it must be isomorphic with L. Thus, $E' \cong E_S(L)$ as claimed.

Proposition. *Let R be a Noetherian ring, and let M be a finitely generated R-module.*
(a) *The support of M is*

$$\bigcup_{j=0}^{\infty} \mathrm{Supp}\,\big(\mathrm{Ext}^j_R(M,\,R)\big).$$

Morever, if

$$N = \max\{\mathrm{depth}(R_P) : P \text{ is minimal in } \mathrm{Supp}\,(M)\},$$

then

$$\mathrm{Supp}\,(M) = \bigcup_{j=0}^{N} \mathrm{Supp}\,\big(\mathrm{Ext}^j_R(M,\,R)\big).$$

(b) *If R is a complete local ring of Krull dimension d, and $\_^\vee$ denotes the dual into the injective hull of the residue class field, then*

$$\mathrm{Supp}\,(M) = \bigcup_{j=0}^{d} \mathrm{Supp}\,\big(H^j_m(M)^\vee\big).$$

Proof. Clearly, if $M_P = 0$ then all of the $\mathrm{Ext}^j_R(M,R)_P$ vanish. It suffices to show that if $M_P \neq 0$ then some $\mathrm{Ext}^j_{R_P}(M_P,\,R_P) \neq 0$ for $j \leq N$. It is enough to show this after localizing further at a minimal prime of $\mathrm{Supp}\,(M)$ that is contained in P. Therefore it suffices to show this when P is a minimal prime of $\mathrm{Supp}\,(M)$. But then we know that the first non-vanishing $\mathrm{Ext}^j_{R_P}(M_P,R_P)$ occurs at the depth of R_P on the annihilator of M_P. Since P is minimal in the support of R, M_P has finite length over R_P, and so its annihilator contains a power of PR_P. Thus, the first non-vanishing $\mathrm{Ext}^j_{R_P}(M_P,R_P)$ occurs when j is the depth of R_P, and so $j \leq N$. This completes the proof of part (a).

For part (b) we may map a complete regular local ring $(S,\,n,\,K) \twoheadrightarrow (R,m,K)$, so that $R \cong S/J$. Then $\mathrm{Spec}\,(R)$ may be identified with $V(J) \subseteq \mathrm{Spec}\,(S)$ as a closed set. We may think of all of the modules involved as S-modules instead, with supports that are subsets of $V(J)$, and the issues are unaffected. But now, since $H^j_m(M) \cong H^j_n(M)$, and since, by the Discussion above, it does not matter whether we take the dual into $E_S(K)$ or $E_R(K)$, we have $H^j_n(M)^\vee \cong \mathrm{Ext}^{s-j}_S(M,\,S)$ for every integer j by local duality, and the result now follows from part (a). □

We also note:

Proposition. *Let R be any complete local ring and let M be any finitely generated R module. Then* $\dim\,(H^i_m(M))^\vee \leq i$ *for all $i \geq 0$, where $\_^\vee$ is the dual into the injective hull of the residue class field.*

Proof. We use induction on the dimension of M. If M has dimension 0, we have that $H^0_m(M)$ has finite length and, therefore, its dual has finite length as well, while for $i > 0$, we have that $H^i_m(M) = 0$. Thus, the result holds.

In the general case, we may take a prime cyclic filtration of M, and by taking the dual of the long exact sequence for local cohomology we may reduce to the case where M is prime cyclic. Since it does not matter over which ring we take the dual, we may replace R by $R/\mathrm{Ann}_R M$, and so assume that $M = R$ is a complete local domain. Then R is module-finite over a regular local ring A, and we may take local cohomology and duals over A instead of over R. Again, we may reduce to the case where R is a prime cyclic A-module, but since we may assume by induction that the result holds for dimension smaller than that of A, we may even assume without loss of generality that $R = A = M$. But then $H^i_m(A) = 0$ unless $i = \dim(A)$, in which case $H^i_m(A)^\vee \cong A$ has dimension i, which suffices. □

We next observe:

Theorem. *Let (R, m, K) be a complete local ring, let $E = E_R(K)$ be an injective hull of the residue class field, and let $\_^\vee$ denote $\mathrm{Hom}_R(\_, E)$. Let N be any finitely generated R-module, and let $E^\bullet$ denote the complex*

$$0 \to E^0 \to E^1 \to \cdots \to E^j \to \cdots,$$

which we assume to be a minimal injective resolution of N. Let $\mathcal{E}^\bullet$ be the complex $H^0_m(E^\bullet)$. Then:

(a) *Each term of $\mathcal{E}^\bullet$ is a finite direct sum of copies of E. In fact, $\mathcal{E}^\bullet$ has the form*

$$0 \to E^{\mu_0} \to E^{\mu_1} \to \cdots \to E^{\mu_j} \to \cdots$$

where $\mu_j = \mu_j(m, N)$.

(b) *Each map $E^{\mu_i} \to E^{\mu_{i+1}}$ is given by a $\mu_{i+1} \times \mu_i$ matrix over R with entries in m.*

(c) *The terms $\mathcal{E}^j$ are 0 for $j < k = \mathrm{depth}_m N$, while $\mathcal{E}^k \neq 0$.*

(d) *The cohomology of the complex $\mathcal{E}^\bullet$ is $H^\bullet_m(N)$.*

(e) *The complex $\mathrm{Hom}_R(E, \mathcal{E}^\bullet)$ is the same as the complex $\mathrm{Hom}_R(E, E^\bullet)$. It is a complex consisting of finitely generated free modules. Its cohomology is $\mathrm{Ext}^\bullet_R(E, N)$. It has the form*

$$0 \to R^{\mu_0} \to R^{\mu_1} \to \cdots \to R^{\mu_j} \to \cdots.$$

where the matrices of the maps are the same as in part (b), and so have entries in m.

(f) *The complex $\mathcal{E}^{\bullet\vee}$ is a complex of finitely generated free modules. Its cohomology consists of the modules $H^\bullet_m(N)^\vee$. It has the form*

$$0 \leftarrow R^{\mu_0} \leftarrow R^{\mu_1} \leftarrow \cdots \leftarrow R^{\mu_j} \leftarrow \cdots.$$

It is the dual into R of the complex displayed in part (e).

Proof. For part (a), note that since $\mathrm{Ass}\big(E(R/P)\big) = \{P\}$, the value of $H^0_m(\_)$ on an injective module consists precisely of the direct sum of those copies of $E(R/P)$ such that $P = m$. Thus, in fact, $H^0_m(E^j) \cong E^{\mu_j}$.

Quite generally, the maps of modules

$$\phi : \bigoplus_{j=1}^{b} V_j \to \bigoplus_{i=1}^{a} W_i \cong \prod_{i=1}^{a} W_i$$

are in bijective correspondence with the $a \times b$ matrices of maps $(\phi_{i,j})$ where, for all i, j, $\phi_{i,j} : V_j \to W_i$. Here, the map ϕ corresponding to the matrix $(\phi_{i,j})$ is such that its value on $v \in V_j$ is

$$(\phi_{1,j}(v), \, \dots \, , \phi_{a,j}(v)).$$

The statement in part (b) that the map is given by a μ_{i+1} by μ_i matrix over R now follows from the fact that $\mathrm{Hom}_R(E,\, E) \cong R$. We already know that, because of the minimality of the resolution, every element in the socle in $E^{\mu_{i+1}}$ is killed by the map: it follows at once that the entries of the matrix are in m.

Part (c) is then clear, since it is equivalent to the assertion that $\mu_j = 0$ for $j < d$, while $\mu_d \neq 0$, and μ_j is the K-vector space dimension of $\mathrm{Ext}^j_R(K,\, N)$. Part (d) follows simply because $H^i_m(\_)$ may be viewed as the ith right derived functor of $H^0_m(\_)$.

The first statement in part (e) follows because every element of the image of map from E is killed by a power of m, and so $\mathrm{Hom}_R(E,\, E^j) \cong \mathrm{Hom}_R\big(E,\, H^0_m(E^j)\big)$. The statement about cohomology then follows from the definition of Ext using injective resolutions of the second module, while the remaining statements are immediate from parts (a) and (b) above and the fact that $\mathrm{Hom}_R(E,\, E) \cong R$.

Part (f) is immediate from the fact that $\mathrm{Hom}_R(E,\, E) \cong R$, part (d), and the fact that on modules Q that are finite direct sums of copies of E, we may identify the functors $Q \mapsto Q^\vee$ and $Q \mapsto \mathrm{Hom}_R\big(\mathrm{Hom}_R(E,\, Q),\, R\big)$. In fact, we have a natural identification of R with $\mathrm{Hom}_R(E,\, E)$, and so it suffices to give a natural isomorphism

$$\theta_Q : Q^\vee \cong \mathrm{Hom}_R\big(\mathrm{Hom}_R(E,\, Q),\, \mathrm{Hom}_R(E,\, E)\big).$$

We can do this as follows. Given $f : Q \to E$ and $g : E \to Q$, let

$$\theta_Q(f)(g) = f \circ g \in \mathrm{Hom}_R(E,\, E).$$

Since this set-up commutes with direct sums $Q = Q_1 \oplus Q_2$, it suffices to verify that θ_Q is an isomorphism when $Q = E$, which is straightforward. $\square$

Theorem. *Let $(R,\, m,\, K)$ be complete local and let N be any finitely generated R-module. Let $d = \mathrm{depth}_m R$. Let E be an injective hull of the residue class field of R. Then $\mathrm{Ext}^i_R(E,\, N) = 0$ for $i < d$.*

Proof. Let $\_^\vee$ denote the functor $\mathrm{Hom}_R(\_,\, E)$. For any R-modules Q and W we have that

$$(*) \quad \mathrm{Ext}^i_R(Q, W^\vee) \cong \mathrm{Tor}^R_i(Q,\, W)^\vee,$$

for if $G_\bullet$ is projective resolution of Q, the latter is

$$H_i(G_\bullet \otimes_R W)^\vee \cong H_i\big(\mathrm{Hom}(G_\bullet \otimes_R W,\, E)\big)$$

since E is injective, and by the adjointness of tensor and Hom this is

$$\cong H^i\Big(\mathrm{Hom}_R\big(G_\bullet,\, \mathrm{Hom}_R(W,\, E)\big)\Big) \cong \mathrm{Ext}^i_R(Q,\, W^\vee).$$

Thus,

$$\mathrm{Ext}^i_R(E,\, N) \cong \mathrm{Ext}^i_R(E, N^{\vee\vee}) \cong \mathrm{Tor}_i^R(E,\, N^{\vee})^{\vee}.$$

Let $W_n = \mathrm{Hom}_R(N/m^nN,\, E)$. Every map from N to E has finitely generated image that is, therefore, killed by a fixed power m^n of the maximal ideal of R, and so factors $N \twoheadrightarrow N/m^nN \to E$. It follows that $N^{\vee} = \varinjlim_n W_n$. Therefore

$$\mathrm{Ext}^i_R(E,\, N) \cong \big(\mathrm{Tor}_i^R(E, \varinjlim_n W_n)\big)^{\vee} \cong \big(\varinjlim_n \mathrm{Tor}_i^R(E, W_n)\big)^{\vee}.$$

Since $\_^{\vee}$ is exact, this may be identified with

$$\varprojlim_n \big(\mathrm{Tor}_i^R(E,\, W_n)\big)^{\vee}.$$

We may think of $\mathrm{Tor}_i^R(E,\, W_n)$ as $\mathrm{Tor}_i^R(W_n,\, E)$ and apply $(*)$ again to get,

$$\mathrm{Tor}_i^R(W_n,\, E)^{\vee} \cong \mathrm{Ext}^i_R(W_n,\, E^{\vee}) \cong \mathrm{Ext}^i_R(W_n,\, R) = 0$$

for $i < d$, since $\mathrm{depth}_m R = d$. $\square$

We shall soon show that under the hypotheses of the Theorem just above, if $\mathrm{id}_R N < \infty$, then $\mathrm{Ext}^d_R(E,\, N)$ is a finitely generated module of finite projective dimension with the same support as N.