Math 711: Lecture of November 28, 2005

Theorem. *Let (R, m, K) be complete local and let N be a finitely generated R-module of finite injective dimension. Let $d = \mathrm{depth}_m R$. Let E be an injective hull of the residue class field of R. Then $M = \mathrm{Ext}^d_R(E, N)$ is a finitely generated module of finite projective dimension equal to $d - \mathrm{depth}_m N$ over R, and such that $\mathrm{Supp}\,(M) = \mathrm{Supp}\,(N)$.*

Proof. We use the same notations as in the Theorem stated on page 3 of the Lecture Notes for November 21. In particular, $\_^\vee = \mathrm{Hom}_R(\_, E)$, $E^\bullet$ is a minimal injective resolution of N, and $\mathcal{E}^\bullet$ is $H^0_m(E^\bullet)$. However, now we know that $\mathrm{id}_R N$ is finite, and must agree with $d = \mathrm{depth}_m R$. By parts (c) and (e) of the Theorem mentioned, $\mathrm{Ext}^\bullet_R(E, N)$ is $H^\bullet\bigl(\mathrm{Hom}_R(E, \mathcal{E}^\bullet)\bigr)$, and $\mathrm{Hom}_R(E, \mathcal{E}^\bullet)$ is a free complex of the form

$$0 \to R^{\mu_k} \to \cdots \to R^{\mu_d} \to 0$$

since $\mu_i = 0$ for $i < k$. By the last Theorem of the Lecture Notes of November 21, the cohomology vanishes except at the R^{μ_d} term. Therefore this complex is a free resolution of its augmentation, which is $M = \mathrm{Ext}^d_R(E, N)$. Moreover, the matrices of the maps have entries in m, so that it is a minimal resolution. We know that $\mu_k \neq 0$, where $k = \mathrm{depth}_m N$, and so $\mathrm{pd}_R M = d - k$, as claimed.

It remains to see that $\mathrm{Supp}\,(M) = \mathrm{Supp}\,(N)$. We know that $\mathrm{Supp}\,(M) = \bigcup_j \mathrm{Ext}^j_R(M, R)$ by part (a) of the Proposition at the top of the second page of the Lecture Notes from November 21. We may use the resolution to compute these Ext modules. But when we apply $\mathrm{Hom}_R(\_, R)$, by part (f) of the Theorem cited, we obtained the complex $(\mathcal{E}^\bullet)^\vee$, and its cohomology is $\bigl(H^\bullet_m(N)\bigr)^\vee$. Therefore $\mathrm{Supp}\,(M) = \bigcup_j \mathrm{Supp}\, H^j_m(N)^\vee$, which by part (b) of the Proposition cited earlier is the same as $\mathrm{Supp}\,(N)$. $\square$

We can now prove:

Theorem (Peskine-Szpiro). *The intersection theorem implies an affirmative answer to Bass's question, i.e., it implies that if a local ring (R, m, K) has a nonzero finitely generated module N of finite injective dimension, then R is Cohen-Macaulay.*

Proof. We use induction on $\dim\,(R)$: the case where $\dim\,(R) = 0$ is trivial. We may assume without loss of generality that R is complete. By the preceding result, R has a finitely generated module M such that $\mathrm{pd}_R M < \infty$ and $\mathrm{Supp}\,(M) = \mathrm{Supp}\,(N)$. We use induction on $\dim\,(R)$. If $\dim\,(R) = 0$ then R is Cohen-Macaulay and we are done.

Choose P, a minimal prime of R, such that $\dim\,(R/P) = \dim\,(R)$. We consider two cases. The first is where $M/PM \cong (R/P) \otimes_R M$ is zero-dimensional, i.e., of finite length. Then $\dim\,(R) = \dim\,(R/P) \le \mathrm{pd}_R M$ (by the intersection theorem) $\le \mathrm{depth}_m R$, i.e., $\dim\,(R) \le \mathrm{depth}_m R$. Since we always have the other inequality, R is Cohen-Macaulay, and the proof in this case is complete.

Now suppose that $\dim\,(M/PM) > 0$. Then there is some prime Q_0 other than the maximal ideal in $\mathrm{Supp}\,(M/PM)$, and we can choose Q prime with $Q_0 \subseteq Q \subset m$ such that

$\dim(R/Q) = 1$, i.e., such that Q is as large as possible with respect to the condition that it be strictly contained in m. Since $Q \in \operatorname{Supp}(M/PM)$, we have that $P \subseteq Q$ and that $Q \in \operatorname{Supp}(M) = \operatorname{Supp}(N)$.

By the second Lemma on the first page of the Lecture Notes of November 21, we have that

$$(1) \quad \dim(R/Q) + \operatorname{depth}(R_Q) = \operatorname{depth}_m R,$$

since $Q \in \operatorname{Supp}(N)$ with $\operatorname{id}_R N < \infty$. Since N_Q is a nonzero module of finite injective dimension over R_Q, we have that R_Q is Cohen-Macaulay by the induction hypothesis. Thus, the displayed equation (1) becomes

$$(2) \quad 1 + \dim(R_Q) = \operatorname{depth}_m R.$$

Since R/P is a complete local domain, we have that

$$\dim(R/P) = \dim(R/Q) + \operatorname{height}(Q/P) = 1 + \dim(R_Q/PR_Q).$$

Now, P was chosen so that $\dim(R/P) = \dim(R)$, and so we have

$$\dim(R) = 1 + \dim(R_Q/PR_Q).$$

Now, R_Q is a Cohen-Macaulay ring, and so all minimal primes yield quotients whose dimension is $\dim(R_Q)$. Therefore, we have

$$(3) \quad \dim(R) = 1 + \dim(R_Q).$$

Comparing (2) and (3), we have that $\dim(R) = \operatorname{depth}_m R$, and so R is Cohen-Macaulay. $\square$

Discussion: The existence of big Cohen-Macaulay modules versus the canonical element conjecture. We have seen that if R has a big Cohen-Macaulay module, then $\eta_R \neq 0$. However, in terms of proving conjectures that do not refer to big Cohen-Macaulay modules directly, so far as I know, every result that follows from the existence of big Cohen-Macaulay modules follows from the canonical element conjecture.

It is therefore tempting to ask whether one can prove that if $\eta_R \neq 0$ for a local ring R, then R has a big Cohen-Macaulay module. So far as I know, this is an open question.

Discussion: Big Cohen-Macaulay modules versus big Cohen-Macaulay algebras. Similarly, there do not seem to be results that follow from the existence of big Cohen-Macaulay algebras that do not follow as well from the canonical element conjecture if one thinks about such algebras "one ring at a time." However, a statement known as the "weakly functorial existence of big Cohen-Macaulay algebras" does yield something new, the vanishing conjecture for maps of Tor (this is a Theorem in the equal characteristic case and a conjecture in the mixed characteristic case). We first want to discuss briefly what is known about the existence of big Cohen-Macaulay algebras, and then how this can be used to prove the vanishing conjecture for maps of Tor in equal characteristic, as well as some consequences of the vanishing conjecture.

Recall that if R is any domain, R^+ denotes the integral closure of R in an algebraic closure of its fraction field, which is unique up to non-unique isomorphism. We may also characterize R^+ as a domain integral over R that does not admit any proper integral extension domain. A third characterization is that R^+ is a domain integral over R such that every monic polynomial over R^+ factors into linear factors over R^+. We leave the verification of the equivalence to the reader.

We want to observe that if $R \to S$ is any map of domains, then there is a commutative diagram:

$$\begin{tikzcd} R^+ \arrow[r] & S^+ \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}.$$

To see this, note that $R \to S$ factors $R \twoheadrightarrow \overline{R} \subseteq S$, where $\overline{R}$ is the image of R in S. The problem of finding the needed map $R^+ \to S^+$ can be solved by filling in both missing arrows in the top row of the diagram

$$\begin{tikzcd} R^+ & \overline{R}^+ & S^+ \\ R \arrow[u] \arrow[r] & \overline{R} \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}.$$

so that the two squares commute. This reduces the problem of finding a suitable map $R^+ \to S^+$ to the separate cases where $R \to S$ is surjective and where $R \subseteq S$. In the latter case, one can simply observe that the integral closure of R in S^+ must be R^+, since an algebraically closed field containing S will contain an algebraic closure of the fraction field of R. In case $R \to S$ is surjective, so that $S = R/P$, note that, since R^+ is integral over R, there is prime ideal Q of R^+ lying over P. It follows that R/P injects into R^+/Q. The latter ring is a domain integral over R/P, since R^+ is integral over R. Moreover, every monic polynomial factors into linear factors over R^+/Q, from which it follows that $R^+/Q \cong (R/P)^+$, and we have the required map $R^+ \twoheadrightarrow R^+/Q \cong (R/P)^+$.

When we speak of the *weakly functorial existence of big Cohen-Macaulay algebras* for a class of local rings, we mean that whenever R and S are local rings in the class and we have a local homomorphism $R \to S$, there exists a commutative diagram:

$$(\#) \quad \begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

such that B is a big Cohen-Macaulay algebra over R and C is a big Cohen-Macaulay algebra over S.

This is known in equal characteristic. The positive characteristic p case was settled by [M. Hochster and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Annals of Mathematics, **135** (1992) 53–89]. There it is shown that if R is an excellent local domain of characteristic $p > 0$, then R^+ is a big Cohen-Macaulay algebra for R. In case R is a complete local domain, these big Cohen-Macaulay algebras are quasi-local. The map between the big Cohen-Macaulay algebras can be filled in by the argument given above. This result also has an analogue in the graded case.

We note that G. Dietz in his thesis [G. Dietz, *Closure operations in positive characteristic and big Cohen-Macaulay algebras*, Ph.D. thesis, University of Michigan, 2005] proves much stronger results in characteristic p:

(1) When $R \to S$ is a local map of complete local domains and B is any big Cohen-Macaulay algebra over R, one can construct a commutative diagram (#) as above in which C is a big Cohen-Macaulay algebra over S.

Moreover:

(2) Any two big Cohen-Macaulay algebras B and C over a complete local domain R both map to a third big Cohen-Macaulay R-algebra, D.

It is also known in equal characteristic 0 that if $R \to S$ is a local homomorphism of complete local domains then there exists a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that B is a big Cohen-Macaulay algebra over R and C is a big Cohen-Macaulay algebra over S. This follows by reduction to positive characteristic, although the argument is quite complicated for one of its kind.

However, the strengthened results (1) and (2) that G. Dietz proved in positive characteristic are open questions in the equal characteristic 0 case: they do not appear to yield readily to reduction to positive characteristic. The "strong" version of the weakly functorial existence of big Cohen-Macaulay algebras is of considerable interest: when it holds, contracted expansion from a big Cohen-Macaulay algebra yields a closure operation that gives a good analogue of tight closure theory.

We next want to discuss the vanishing conjecture/theorem for maps of Tor.

Conjecture. *Let $A \to R \to S$ be maps of Noetherian rings such that A is regular, R is module-finite and torsion-free over A, and S is regular. Let M be any A-module. Then the map* $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ *is 0 for all $i \geq 1$.*

We shall show that this follows from the weakly functorial existence of big Cohen-Macaulay algebras. Thus, it is theorem rather than a conjecture for rings containing a field.