Math 711: Lecture of November 30, 2005

In considering the vanishing conjecture for maps of Tor, we note the following easy reductions in the problem.

(1) As already mentioned, we may assume that M is finitely generated as an A-module.

(2) If there is a nonzero element u in the image of $\mathrm{Tor}_i^A(M,\,R) \to \mathrm{Tor}_i^R(M,\,S)$, we can choose a prime $\mathcal{M}$ of S in the support of Su. Then the image of u in $\mathrm{Tor}_i A(M,\,S_{\mathcal{M}}) \cong \mathrm{Tor}_i^A(M,\,S)_{\mathcal{M}}$ is nonzero, and we may replace S by $S_{\mathcal{M}}$. Henceforth, we assume that S is local.

(3) We may the replace S by $\widehat{S}$, which is faithfully flat over S. Note that $\mathrm{Tor}_i^A(M,\,\widehat{S}) \cong \mathrm{Tor}_i^A(M,\,S) \otimes_S \widehat{S}$.

(4) Let m be the contraction of $\mathcal{M}$ to A. Then we may replace A by A_m, M by M_m, and R by R_m. Note that, because the elements of $A - m$ are invertible when mapped into S, $\mathrm{Tor}_i^A(M,\,S) \cong \mathrm{Tor}_i^{A_m}(M_m,\,S)$. Henceforth, we assume that A is local.

(5) Instead of using A, M, R, and S, we may use $\widehat{A}$, $\widehat{M} \cong \widehat{A} \otimes_A M$, $R_1 = \widehat{A} \otimes_A R$, and S, noting that we have maps $\widehat{A} \to R_1 \to S$ because S is complete. Observe that the fact that R is torsion-free over A is equivalent to the fact that R can be emebedded in a finitely generated free A-module, and it is then clear that R_1 can be embedded in a finitely generated free $\widehat{A}$-module. Henceforth, we assume that A and T are complete local rings.

(6) Since A is normal and R is torsion-free and module-finite over A, the going-down theorem holds for $A \to R$. Let $\mathcal{Q}$ be the kernel of the map $R \to S$ and let $\mathcal{P}$ be the contraction of $\mathcal{Q}$ to A. Then there is a prime $\mathcal{Q}_0$ of R contained in $\mathcal{Q}$ and lying over 0 in A, and we have maps $A \to R \to R/\mathcal{Q}_0 \to S$. It follows that we may replace R by $R/\mathcal{Q}_0$, which is a domain module-finite over A. Thus, in considering the vanishing conjecture for maps of Tor, it suffices to consider the case where A, R, and S are complete local domains and R is a module-finite extension of R, while A and T are regular.

We next want to indicate two proofs of the vanishing conjecture in characteristic $p > 0$.

Proof of the vanishing conjecture for maps of Tor using tight closure theory. Since the localization of R at $A - \{0\}$ is a finite dimensional vector space over A, we can find a finitely generated A-free module $W \subseteq R$ and an element $c \in A - \{0\}$ such that $cR \subseteq W$. Now let $G_\bullet$ be a free resolution of M by finitely generated free modules, let $i \geq 1$ be an integer, and let z be a cycle in $G_i \otimes_A R$, so that z represents an element of $\mathrm{Tor}_i^A(M,\,R)$. We shall show that z is in the tight closure, over R, of the module of boundaries B within $G_i \otimes_A R$. First note that $F_R^e(G_\bullet \otimes_A R) \cong F_A^e(G_\bullet) \otimes_A R$. [In one case we first map the entries of the matrices to R and then raise them to the p^e th power, while in the other we raise them to the p^e th power and then map them to R. This can also be said in a functorial way. Let R' (respectively, A') denote R (respectively, A) as an algebra over itself using the endomorphism F^e. Then the diagram

$$\begin{array}{ccc} A' & \longrightarrow & R' \\ \uparrow & & \uparrow \\ A & \longrightarrow & R \end{array}$$

commutes (both vertical maps are induced by F^e), and the statement that we need is simply that $R' \otimes_{A'} (A' \otimes_A \_\,)$ may be naturally identified with $R' \otimes_R (R \otimes_A \_\,)$. By the associativity of tensor, both are $R' \otimes_A \_\,$.] But $F^e_A(G_\bullet) \otimes_A R$ has the subcomplex $F^e_A(G_\bullet) \otimes_A W$, which is acyclic: since F is exact over a regular ring, $F^e_A(G_\bullet)$ is acyclic, and W is A-free. Now cz^q is a cycle, and is in $F^e_A(G_\bullet) \otimes W$, since $cR \subseteq W$, and so cz^q is in $B^{[p^e]}$ in $F^e_R(G_i \otimes R)$. This shows that $z \in B^*$. But then, when we apply $\_\,{}_R S$, we find that the image of z becomes a boundary: it is in the tight closure of the boundaries, and over the regular ring S, every finitely generated submodule of every module is tightly closed. $\square$

We can replace the condition that S be regular by the condition that each of its completed local rings is weakly F-regular. In the locally excellent case, this is simply the condition that S be F-regular. There are several other ways to generalize, but we do not want to get too technical here.

Before giving the second proof, we need a preliminary result.

Proposition. *Let S and A be rings.*

(a) *If (S, n, L) is regular local, an S-module C is faithfully flat if and only if C is a big Cohen-Macaulay module for S.*
(b) *If (S, n, L) is an Artin local ring, an S-module C is flat if and only if C is free.*
(c) *If C is faithfully flat over the local ring (S, n, L) with $u \in C - nC$ and we map $S \to C$ so that $1 \mapsto u$, then the map $S \to C$ is pure, i.e., it is injective and remains so when we tensor over S with any S-module N.*
(d) *If $C_0 \to C$ is a pure map of A modules, and $B \subseteq G$ are modules, then the intersection of the image of $B \otimes_A C$ in $G \otimes_A C$ with the image of $G \otimes_A C_0$ is the image of $B \otimes_A C_0$ in $G \otimes_A C_0$.*
(e) *If $C_0 \to C$ is a pure map of A-modules, and $G_\bullet$ is any complex of A-modules, then for all i the map $H_i(G_\bullet \otimes C_0) \to H_i(G_\bullet \otimes C)$ is injective.*
(f) *If $A \to S$ is a ring homomorphism, M is an A-module, and $C_0 \to C$ is pure over S, then $\mathrm{Tor}^A_i(M, C_0) \to \mathrm{Tor}^A_i(M, C)$ is injective for all i.*

Proof. For part (a), "only if" is clear. For "if," it will suffice to show that for every prime P of S and for all $i \geq 1$, $\mathrm{Tor}^S_i(S/P, C) = 0$. This is clearly true for all $i \gg 0$, since S is regular local. Assume that the result is true for all integers $> i$, where $i \geq 1$. We prove it for i. Let $x_1, \dots, x_h$ be a maximal S-sequence in P. Then P is an associated prime of $(x_1, \dots, x_h)R$, and we have a short exact sequence $0 \to S/P \to S/(x_1, \dots, x_h)S \to Q \to 0$ for some Q. The long exact sequence for Tor yields the exactness of

$$\cdots \to \mathrm{Tor}^S_{i+1}(Q, C) \to \mathrm{Tor}^S_i(S/Q, C) \to \mathrm{Tor}^S_i(S/(x_1, \dots, x_h)S, C) \to \cdots$$

Q has a filtration by prime cyclic modules and so the leftmost term vanishes. The rightmost term vanishes because $x_1, \ldots, x_h$ is a regular sequence on C. Thus, the middle term vanishes as well, as required.

For part (b) (note that we are not assuming that C is finitely generated), first observe that we have the analogue of Nakayama's lemma for any module D, without finiteness hypothesis. For if $D = nD$, then $D = n^t D$ for all t, and for $t \gg 0$, $n^t = 0$. Choose a vector space basis for C/nC, and lift it to a set of elements for in $u_j \in C$. Choose a free S-module G with a free basis b_j in bijective correspondence with the u_j, and map $G \to C$ sending $b_j \mapsto u_j$ for every j. Then $(S/n) \otimes_S (C/\mathrm{Im}\,(G)) = 0$, and so $\mathrm{Im}\,(G) = C$, and we have a short exact sequence $0 \to N \to G \to C \to 0$ for some N. Since $\mathrm{Tor}^1_S(L, C) = 0$, the sequence remains exact when we apply $L \otimes_S \_$. Since $L \otimes G \to L \otimes C$ is an isomorphism by our construction of G, we have that $L \otimes_S N = 0$, and so $N = 0$.

For part (c), it suffices to show that the map $N \to N \otimes_S C$ is injective for every finitely generated module N. If $u \in N$ maps to 0, we can choose $t \gg 0$ such that $u \notin N/n^t N$. Therefore, we obtain a counterexample with N killed by n^t. It therefore suffices to prove that the map is pure after we apply $S/n^t S \otimes_S \_$: we can then tensor with N over $S/m^t S$. The hypothesis of flatness and that $u \notin nC$ are preserved by the base change. Thus, we have reduced to the case where S is an Artin local ring. But then C is free, and the element u is part of a free basis, so that $S \mapsto C$ splits.

For part (d), note that we have $G \otimes_A C_0 \subseteq G \otimes_A C$. Moreover, $(G/B) \otimes_A C_0$ injects into $(G/B) \otimes C$, which gives exactly what we need.

For part (e), let B be the image of G_{i-1} in $G_i = G$. If $z \in G \otimes_A C_0$ is a cycle that maps to 0 in $H_i(G_\bullet \otimes_A C)$ then in $G \otimes_A C$ it is a boundary and, hence, in the image of $B \otimes_A C$ intersected with the image of $G \otimes_A C_0$. But then, by part (d), it is in the image of $B \otimes_A C_0$, and so represents 0 in $H_i(G_\bullet \otimes_A C_0)$.

For part (f), we observe that if $C_0 \to C$ is pure over S then it is pure over A. For if M is any A-module, $M \otimes_A \_$ is the same as $(M \otimes_A S) \otimes_S \_$. Thus, we may assume that $C_0 \to C$ is pure over A. The result is now immediate from part (a) applied to $G_\bullet$, a projective resolution of M over A. $\square$

Remark. Here is another approach to the proof of part (e). The result is obvious when C_0 is a direct summand of C as an A-module. But we can reduce to this case. We can write C as $(C_0 \oplus P)/Q$ where P is a free module and Q is a suitable submodule meeting C_0 in 0. Consider all maps $C_0 \to C_1$ where C_1 has the form $(C_0 \oplus P_1)/Q_1$ where P_1 is the finitely generated submodule of P generated by a finite subset of the free basis, and Q_1 is some finitely generated submodule of $Q \cap (C_0 \oplus P_1)$. Then $C_0 \to C$ is the direct limit of the maps $C_0 \to C_1$ as C_1 varies. Each $C_0 \to C_1$ is evidently pure, and each cokernel C_1/C_0 is finitely presented by construction. But then C_0 is a direct summand of C_1, and the injectivity of the map of complexes is clear. The stated result now follows by taking a direct limit. $\square$

Proof of the vanishing conjecture for maps of Tor using the weakly functorial existence of big Cohen-Macaulay algebras. In fact, while the arguments in the literature use algebras, we shall carry the argument through using big Cohen-Macaulay modules instead. We need the following statement: given a local map of complete local domains $(R, m, K) \to (S, n, L)$ of characteristic p, there exist big Cohen-Macaulay modules B over R and C over S and an R-linear map $B \to C$ such that the image of B is not contained in nC. In this circumstance, we can pick $u \in C - nC$ such that u is the image of an element $y \in B$.

This yields a commutative diagram:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

where the vertical arrow on the left sends $1 \mapsto y$ and the vertical arrow on the right sends $1 \mapsto u$. Note that if we have weakly functorial big Cohen-Macaulay algebras, we may use these for B and C in the diagram, with $y = 1$ in B and $u = 1$ in C. We then get a commutative diagram:

$$\begin{array}{ccc} \mathrm{Tor}_i^A(M,\, B) & \longrightarrow & \mathrm{Tor}_i^A(M,\, C) \\ \uparrow & & \uparrow \\ \mathrm{Tor}_i^A(M,\, R) & \longrightarrow & \mathrm{Tor}_i^A(M,\, S) \end{array}$$

Now B is a big Cohen-Macaulay algebra over R and every system of parameters for A is a system of parameters for R, so that B is a big Cohen-Macaulay module over A. Since A is regular, part (a) of the Proposition shows that B is A-flat. Therefore, the Tor module in the upper left corner of the diagram is 0, and so the composite maps from

$$\mathrm{Tor}_i^A(M,\, R) \to \mathrm{Tor}_i^A(M,\, C)$$

are 0. The map $S \to C$ is pure by part (c) of the Proposition, and so the vertical arrow on the right is injective by part (f). It follows that the arrow on the bottom is 0. $\square$

Remark. An alternative proof of the injectivity of the map

$$\mathrm{Tor}_i^A(M,\, S) \to \mathrm{Tor}_i^A(M,\, C)$$

can be based on the fact that every flat module is a direct limit of finitely generated free modules. Suppose we think of $C = \varinjlim_j C_j$ where the C_j are free. Then some C_j contains an element v that maps to u in C. We call the index j_0 and work only with $j \geq j_0$. We then have a map $S \to C_j$ for all $j \geq j_0$. In each case, the image of 1 is not in nC_j, or else u would be in nC. But then the image of 1 is part of a free basis for C_j, so that $S \to C_j$ is split over S (hence, over A as well), and so the induced map of Tor modules is injective. $\square$

The vanishing conjecture for maps of Tor is a statement that can be formulated in terms of equations with a constraint that certain elements be parameters, and so the equal characteristic 0 case follows from the positive prime characteristic p case.

Thus, the vanishing conjecture for maps of Tor is a theorem in the equal characteristic case. We want to show next that it is a very powerful tool. We therefore explain how it can be used to prove several rather subtle results. In particular, we show how it can be used to prove:

(1) The direct summand conjecture.

(2) The conjecture that direct summands of regular rings are Cohen-Macaulay.

Both of these conjectures remain open in mixed characteristic.

To see how the direct summand conjecture follows, we note that is suffices to prove that if $x_1, \dots, x_d$ is a system of parameters of a complete regular local ring A and R is a module-finite extension domain of A, then we do not have a relation

$$\sum_{j=1}^{d} r_j x_j^{t+1} - (x_1 \cdots x_d)^t = 0$$

in R. Consider the A-module A/I, where

$$I = (x_1^{t+1}, \dots, x_d^{t+1}, (x_1 \cdots x_d)^t)A.$$

We take $S = R/m$, the residue class field of R. Then the map

$$\mathrm{Tor}_1^A(A/I, R) \to \mathrm{Tor}_1^A(A/I, K)$$

is supposed to be injective. For any A-algebra B,

$$\mathrm{Tor}_1^A(A/(f_1, \dots, f_k)A, B)$$

consists of the relations of $f_1, \dots, f_k$ with coefficients in B modulo the relations spanned by the relations on $f_1, \dots, f_k$ over A. By hypothesis, we have a relation $(r_1, \dots, r_d, -1)$ on

$$x_1^{t+1}, \dots, x_d^{t+1}, (x_1 \cdots x_d)^t$$

whose image $(0, \dots, 0, -1)$ in $\mathrm{Tor}_1^A(R/I, K)$ is nonzero, since all relations on

$$x_1^{t+1}, \dots, x_d^{t+1}, (x_1 \cdots x_d)^t$$

over A have coefficients in the maximal ideal of A.

This contradiction proves the direct summand conjecture.

To see how one may deduce that direct summands of regular rings are Cohen-Macaulay, suppose that R is a direct summand of S, where S is regular. It suffices to show that each local ring of R is regular, and by tensoring $R \to S$ with a suitable local ring of R we may assume that (R, m, K) is local. Now replace R by $\widehat{R}$ and S by its completion in the

mS-adic topology. Thus, we may assume that R is complete, and so module-finite over a regular local ring A. Then we have $A \to R \to S$, and with $M = K$ we have that the map

$$\mathrm{Tor}_i^A(K,\, R) \to \mathrm{Tor}_i^A(K,\, S)$$

is 0 for all $i \geq 1$. But since $R \to S$ splits over R (we only need the splitting over A) it follows that the map of Tor modules is injective. Hence, $\mathrm{Tor}_i^A(K,\, R) = 0$ for all $i \geq 1$. Since this is simply the Koszul homology of R with respect to a regular system of parameters for A, it follows that R is Cohen-Macaulay. $\square$

One of our goals will be to show that the vanishing conjecture for maps of Tor is equivalent to the following conjecture (and also to the complete local domain case of the following conjecture):

Conjecture (strong direct summand conjecture). *Let $A \to R$ be a module-finite extension of the regular ring A, where R is torsion-free over A, and let $Q \subseteq R$ be a prime lying over a height one prime P of A such that A/P is regular. Then P is a direct summand of Q as A-modules.*

The issue of splitting is local on A. So we may assume that A is a regular local domain. We may also pass to the completion of A. There will be a minimal prime contained in Q lying over (0) in A, and by killing it we may assume that R is a domain. Once A is local, we may assume that it is generated by a regular parameter x. Then $P = xA \subseteq xS \subseteq Q$, and since $xA \subseteq Q$ splits over A, we have tha $xA \subseteq xS$ splits over A. But this map is simply isomorphic with the inclusion $A \subseteq S$. This shows that the strong direct summand conjecture does imply the direct summand conjecture.