Math 711: Lecture of December 2, 2005

Example. The following example shows that one needs to assume that the height one prime Q of R contracts to a prime ideal of A generated by an element $x \in A$ such that A/XxA is regular. $A = K[[u,v]] = K[[s^2,\, t^3]] \subseteq K[[s,t]]$. $Q = s - t$. The contraction is $u^3 - v^2$. Claim: $A(u^3 - v^2) \subseteq (s-t)R$ does not split. This is, up to isomorphism, the inclusion of $A \to R$ that sends 1 to $(s^6 - t^)/(s-t) = s^5 + s^4t + s^3t^2 + s^2t^3 + st^4 + t^5$, which is in $(s^2, t^3)R = (u,v)R$.

One other reduction may seem odd, but is important here: we may replace R by $A+Q \subseteq R$. This is a ring trapped between A and R, and so is a module-finite extension of A, Q has the same contraction to A as before, and the issue of whether $P \subseteq Q$ splits over A certainly has not changed. Note that once A is regular local, $P = xA$ will be principal, and the hypothesis that A/xA is regular says that x is a regular parameter. The statement that xA splits from Q implies that xA splits from $xR \subseteq Q$, and hence that A is a direct summand of R as an A-module. So the strong direct summand conjecture implies the direct summand conjecture. We shall further explain the connection to the vanishing theorem later.

We next want to reduce to the case where $i = 1$. The point is simply that if $i > 1$ and we take a short exact sequence

$$0 \to M' \to G \to M \to 0,$$

where G is a finitely generated free module, so that M' is a first module of syzygies of M, we have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Tor}^A_{i-1}(M',\, R) & \longrightarrow & \mathrm{Tor}^A_{i-1}(M',\, S) \\
\uparrow & & \uparrow \\
\mathrm{Tor}^A_i(M,\, R) & \longrightarrow & \mathrm{Tor}^A_i(M,\, S)
\end{array}.$$

This uses the functoriality of the long exact sequence for Tor. Thus, by repeatedly changing M, we eventually reduce to the case where $i = 1$.

We next want to reduce to the case where $M \cong A/I$ is a cyclic module. In this reduction we change the rings, and we may lose that the rings are complete local. But we can get back to that case afterward. Again, we map a finitely generated free A-module G onto M. There is an induced map of symmetric algebras $A' = S_A(G) \twoheadrightarrow S_A(M)$. Replace R by $R' = S_R(R \otimes_A G)$ and S by $S' = S_S(S \otimes_A G)$. These are all polynomial rings in the same variables over the original A, R, and S, and we still have $A' \to R' \to S'$. Suppose we know that $\mathrm{Tor}^{A'}_i(S_A(M), R') \to \mathrm{Tor}^{A'}_i(S_A(M), S')$ is 0. On A'-modulules, the functors $\_ \otimes_{A'} R'$ and $\_ \otimes_A R$, may be identified, and we find that $\mathrm{Tor}^{A'}(\_,\, R')$ may be identified with $\mathrm{Tor}^A_\bullet(\_,\, R)$ on A'-modules. Similarly, on A'-modules the functors $\_ \otimes_{A'} S'$ and

$\otimes_A S$ may be identified, and we likewise obtain that $\mathrm{Tor}_\bullet^{A'}(\_\,, S')$ and $\mathrm{Tor}_\bullet^{A}(\_\,, S)$ may be identified on A'-modules. We therefore get that the map

$$\mathrm{Tor}_i^A(S_A(M), R) \to \mathrm{Tor}_i^A(S_A(M), S)$$

is 0. Since $S_A(M)$ is a direct sum of A-modules (where the summands are indexed by degree) whose degree one component is M, we find that

$$\mathrm{Tor}_i^A(M,\, R) \to \mathrm{Tor}_i^A(M,\, S)$$

is 0. We have therefore reduce to the case where M is a cyclic module. We may now carry through the reductions to the case where $A \to R \to S$ are complete local domains again: these reductions do not affect the fact that M is cyclic.

Moreover, we can reduce to the case where $A \to S$ is surjective. To do so, pick a coefficient ring V for A. We have $V \to A \to S$. We focus on the case of mixed characteristic: other cases are easier. Map a complete unramified regular local ring $T \twoheadrightarrow S$. One can lift $V \to S$ to a map $V \to T$. Cf. [L. Avramov, H.-B. Foxby, B. Herzog, *Structure of local homomorphisms* J. Algebra **164** (1994), pp. 124–145]. We shall replace A by $A' = A \widehat{\otimes}_V T$: it maps onto T, and is regular: if one kills a regular system of parameters in A, one gets T/pT, where p is the generator of the maximal ideal in V. This ring is regular, and so A' is regular. Let $R' = R \widehat{\otimes}_V T$. Then R' is module-finite over A', and embeds in a finitely generated free module over A'. Let $M' = A' \otimes_A M$. We have maps $A' \to R'$ and $R' \to S$. Even the map $A' \to S$ is surjective, so that $R' \to S$ is surjective as well.

Then A' is faithfully flat over A, and

$$A' \otimes_A \mathrm{Tor}_\bullet^A(M,\, R) \cong \mathrm{Tor}_\bullet^{A'}(M',\, R')$$

while

$$A' \otimes_A \mathrm{Tor}_\bullet^A(M,\, S) \cong \mathrm{Tor}_\bullet^{A'}(M',\, S),$$

since, on A-modules, $(A' \otimes_A \_\,) \otimes_{A'} S \cong \_ \otimes_A S$. Since S is regular, $S = A/(x_1, \ldots, x_k)$ where the x_j are part of a regular system of parameters.

Therefore, we may assume that A is complete regular local, that R is a domain module-finite over A, and that S is complete and is A/P where $P = (x_1, \ldots, x_h)A$ is generated by part of a regular system of parameters. We may also assume that the map $R \to S$ is surjective. Let Q be the kernel of this map. Since $R/Q \cong A/P$, we must have that $R = A + Q$.

Therefore, we need to understand what it means for the map from $\mathrm{Tor}^1(A/I, R)$, where $R = A + Q$ a domain, to $\mathrm{Tor}^1(A/I, R/Q)$, where $R/Q = A/P$ is regular, to vanish. We shall see that this turns out to mean that $I \cap IQ = IP$ for every ideal I of A. (Note that because I is an ideal of A and Q is an ideal of R that does not contain 1, I and IQ are ordinarily incomparable.) We give this result as follows:

Lemma. *Let $I = (i_1, \ldots, i_k)$ be an ideal of the ring A. Suppose that $A \subseteq R$, another ring, and that Q is an ideal of R such that $Q \cap A = P$ and the induced injection $A/P \to R/Q$ is an isomorphism. Then the map $\theta : \mathrm{Tor}_1^A(A/I, R) \to \mathrm{Tor}_1^A(A/I, R/Q)$ is 0 if and only if $I \cap IQ = IP$.*

Proof. First note that since $A/P \cong R/Q$, every element of R can be represented modulo Q by an element of A, i.e., $R = A + Q$. Also note that the inclusion $IP \subseteq I \cap IQ$ is obvious, so that the issue is whether $I \cap IQ \subseteq IP$.

Now elements of $\mathrm{Tor}^1(A/I, R)$ correspond to relations on $i_1, \ldots, i_k$ of I with coefficients in $R = A+Q$, modulo the span of the relations holding over A: suppose that we have such a relation

$$(*) \quad (a_1 + q_1)i_)1 + \cdots + (a_k + q_k)i_k = 0,$$

where the $a_j \in A$ and the $q_j \in Q$. Then we have that

$$(**) \quad a_1 i_1 + \cdots + a_k i_k = -(q_1 i_1 + \cdots + q_k i_k) \in I \cap Q.$$

Note that every element of $I \cap IQ$ occurs on the two sides of an equation of the form of $(**)$. The image of the relation displayed above in

$$\mathrm{Tor}_1^A(A/I, R/Q) = \mathrm{Tor}_1^A(A/I,\, A/P)$$

is given by the relation

$$(\overline{a_1}, \ldots, \overline{a_k})$$

on the i_j over A/P, where $\overline{a_j}$ is the image of a_j in R/P. The condition for this to be 0 is that it be the image of a relation in A, i.e., that there exist elements $a'_j \in A$ such that

$$a'_1 i_1 + \cdots + a'_k i_k = 0$$

and $a_j \equiv a'_j \bmod P$ for all j, and then

$$\sum_{j=1}^{k} a_j i_j = \sum_{j=1}^{k} (a_j - a'_j) i_j \in IP.$$

In fact, the a'_j exist if and only if

$$\sum_{j=1}^{k} a_j i_j \in IP,$$

for in that case we may write

$$\sum_{j=1}^{k} a_j i_j = \sum_{j=1}^{k} b_j i_j$$

with the $b_j \in P$, and then we may take $a'_j = a_j - b_j$. Since, as already mentioned, every element of $I \cap IQ$ occurs on the two sides of an equation of the form $(**)$, we have that θ vanishes if and only if $IQ \cap I \subseteq IP$, as required. $\square$