Math 711: Lecture of December 5, 2005

We have reduced the study of the vanishing conjecture to the case where $A \to R \to S$ are complete local domains with A and S regular such that $A \to S$ (and, hence, $R \to S$) is surjective, and such that R is module-finite over S. If P and Q are the respective kernels of the maps $A \to S$ and $R \to S$, we also know that P is generated by part of a regular system of parameters for A and, moreover, that $R = A + P$. Beyond that, we may assume that $M = A/I$, and then the vanishing conjecture asserts that $I \cap IQ = IP$.

We next want to reduce to the case where P has just one generator. In the course of this reduction, we lose the hypothesis that the the rings are complete local, but we can get back to that case afterward. The method we use to reduce to the case where P has just one generator is to replace A by the second Rees ring $A[Pt, v] \subseteq A[t, 1/t]$, where $v = 1/t$, and R by $R' = R[PRt, v]$: this is a homomorphic image of $R \otimes_A A'$ and is therefore still module-finite over A'. The quotient of A' by vA' is $\mathrm{gr}_P A$, a polynomial ring over A/P, and is still regular. The quotient of R' by vR' is

$$R/PR \oplus PR/P^2R \oplus \cdots \oplus P^jR/P^{j+1}R \oplus \cdots,$$

and this is an $\mathbb{N}$-graded ring. In this ring, $R/PR - Q/PR$ is a multiplicative system, disjoint from the expansion of Q/PR, and Q/PR becomes nilpotent if we localize at this multiplicative system because Q is a minimal prime of PR. There is therefore a minimal prime of vR' whose intersection with R is Q, and which is necessarily graded. Call this ideal Q'. Working with Q' and vA', we have from the vanishing conjecture for the rings

$$A' \to A' + Q' \twoheadrightarrow (A' + Q')/Q' (\cong A'/vA')$$

that for all ideals of I of A, with $I' = IA'$, $I'Q' \cap I' = I'vA'$ or $IQ' \cap IA' = IvA'$. This means that $IQ \cap A \subseteq (IvA[Pt, v])_0 = IP$. It follows that the general case of the vanishing conjecture reduces to the case where I is principal. We have lost the condition that the rings A, R, and S be local and complete, but we may repeat the earlier argument to reduce to this case again.

Thus, the vanishing conjecture for maps of Tor is equivalent to the conjecture that when Ax is a principal prime of the complete regular local ring A such that A/xA is regular and Q is a height one prime of a local domain R that is a module-finite extension of A such that $R = A + Q$, then for every ideal I of A, $IQ \cap I = Ix$.

We next note that if W is a module over a regular local ring A and $w \in W$, then condition that the map $A \to W$ sending $1 \mapsto w$ split is that for every ideal I of A, the contraction of IW to A is I. (The condition is clearly necessary, for if $g : W \to A$ is a splitting, and $b \mapsto bw \in IW$, then applying g yields that

$$b = g(bw) \in g(IW) \subseteq Ig(W) \subseteq IA = I.$$

For the converse, we show that if $x_1, \ldots, x_d$ is a regular system of parameters for A, it suffices for all of the ideals $I_t = x_1^t, \ldots, x_d^t$ to be contracted. Both this condition and the splitting condition are unaffected by completion. The condition yields that

$$A/I_t \to W \otimes_A (A/I_t) \cong W/I_tW$$

is injective for all t, and we may take a direct limit to conclude that $E \to W \otimes_A E$ is injective, where $E = E_E(K)$ is the injective hull of the residue class field K of A. This yields that $\mathrm{Hom}_A(W \otimes_E, E) \to \mathrm{Hom}_A(E, E)$ is surjective, and, by the adjointness of tensor and Hom, that

$$\mathrm{Hom}_A\big(W, \mathrm{Hom}_A(E, E)\big) \to \mathrm{Hom}_A(E, E)$$

is surjective. By Matlis duality, this means that $\mathrm{Hom}_A(W, A) \to A$ as surjective, as required.)

We can now complete the proof of the equivalence of the vanishing conjecture and the strong direct summand conjecture. Since both imply the direct summand conjecture, we may assume that the direct summand conjecture holds, and this implies that for every ideal I of A, $IQ \cap A \subseteq IR \cap A \subseteq I$, and therefore $IQ \cap A = IQ \cap I$. But since $IQ \cap A \subseteq Q \cap A = Ax$, we also have that $IQ \cap A = IQ \cap Ax = IQ \cap I$.

Now, Ax splits from Q if and only if $IQ \cap Ax = Ix$ for all I. By the calculation in the preceding paragraph, this is equivalent to the condition that $IQ \cap I = Ix$ for every I, and we have already seen that this is equivalent to the vanishing conjecture for maps of Tor. $\square$

There has been a partially successful "metaconjecture" that asserts that a result about regular rings should generalize to arbitrary Noetherian rings if the assumption of regularity is replaced by the hypothesis that certain modules have finite projective dimension.

One can generalize the vanishing conjecture for maps of Tor in this way. Instead of assuming that A is regular, one assumes that M is a Noetherian module of finite projective dimension over A, and that ideals of depth at least k in A have height at least k in R modulo every minimal prime of R. Under these hypotheses and rather weak conditions on the rings, one can show that if $G_\bullet$ is a finite projective resolution of M over R, then the cycles are in the tight closure of the boundaries in the complex $G_\bullet \otimes_A R$ in degree one or more. This implies that cycles become boundaries when one tensors further with a weakly F-regular ring. However, we shall not attempt to give the best such result here, but rather refer the reader to [M. Hochster and C. Huneke, *Phantom Homology*, Memoirs of the Amer. Math. Soc. Vol. **103** Number 490, 1993, Amer. Math. Soc., Providence, R.I.], Theorem 4.13 (the case where $T = S$ considerably generalizes the result we discussed here).

From the point of view of this "metaconjecture," it is reasonable to ask whether, when R is any Noetherian ring, a module-finite algebra extension $R \hookrightarrow S$ such that $\mathrm{pd}_R S < \infty$ has the property that $R \to S$ splits. This question was raised by J. Koh in his thesis [J. H. Koh, *The direct summand conjecture and behavior of codimension in graded extensions*, Ph.D. Thesis, University of Michigan, 1983]. The result is true when R contains a field of characteristic 0, but false in general otherwise: counter-examples were given in Juan

Vélez in characteristic 2 and in mixed characteristic 2: cf. [J. D. Velez, *Splitting results in module-finite extension rings and Koh's conjecture* J. Algebra **172** (1995), pp. 454–469].

The proof in the equal characteristic 0 case depends on developing a notion of trace. We want to assign a trace to every endomorphism of every finitely generated R-module M of finite projective dimension, at least when R is a Noetherian ring such that $\mathrm{Spec}\,(R)$ is connected.

We begin with the case where M is free. In this case, the trace $\mathrm{Tr}_R(f) = \mathrm{Tr}(f)$ of $f : M \to M$ may be defined as the sum of the diagonal entries of a matrix representing f. The matrix depends on the choice of a free basis for M, but the trace does not, because change of basis corresponds to replacing the matrix A by UAU^{-1}, and the trace does not change. It is clear that $f \mapsto \mathrm{Tr}(f)$ is an R-linear map whose value on the identity map is the image of the rank of M in R.

In case R is Noetherian with $\mathrm{Spec}\,(R)$ connected and $\mathrm{pd}_R M < \infty$, we can define an R-linear trace map $\mathrm{Hom}_R(M,\,M) \to R$ as follows. Let W be the multiplicative system of all nonzerodivisors in R. Consider the induced endomorphism $W^{-1}M \to W^{-1}M$ over $W^{-1}R$. Then $W^{-1}R$ is a semilocal ring whose localization at each maximal ideal has depth 0. Since a module of finite projective dimension over a local ring of depth 0 must be free, it follows that $W^{-1}M$ is locally free over $W^{-1}R$. We want to see that the rank is constant. Fix a finite projective resolution $G_\bullet$ of M by finitely generated projective modules. The rank of M_P is the alternating sum of the ranks of the free modules in $(G_\bullet)_P$. Because $\mathrm{Spec}\,(R)$ is connected, every projective module is locally free of constant rank. It follows that the rank of M_P is independent of P for P corresponding to a maximal ideal of $W^{-1}R$. Therefore, $W^{-1}M$ is free over $W^{-1}R$, and we can define the trace of $g \in \mathrm{Hom}_R(M,\,M)$ as the trace of the map induced by g from $W^{-1}M \to W^{-1}M$. The difficulty is that we want the trace to be in $R \subseteq W^{-1}R$, not merely in $W^{-1}R$.

We can show that our trace is in R as follows. The map $f : M \to M$ lifts to a map $\phi_\bullet : G_\bullet \to G_\bullet$. We shall show that $\mathrm{Tr}(f)$ is the alternating sum $\sum_{j=0}^{h}(-1)^j \mathrm{Tr}_R(\phi_j)$, where h is the length of $G_\bullet$. Since the alternating sum is evidently in R, this will prove what we need. It suffices to prove the equality after localization at W. Therefore, we need only prove the result that if we have a map from an exact (not merely acyclic) complex of finitely generated free modules

$$0 \to F_k \to \cdots \to F_0 \to 0$$

to itself, the alternating sum of the traces of the maps is 0 (the F_i correspond to the localizations of the G_i at W, the base ring is now $W^{-1}R$, and $W^{-1}M$ is now included). If the complex has length at most three, say $0 \to F_2 \to F_1 \to F_0 \to 0$ (where some of these may be 0), we have that $F_1 \cong F_2 \oplus F_0$. We can choose free bases for F_2 and F_0, and their union will be a free basis for F_1. The matrix of ϕ_1 then has block form

$$\begin{pmatrix} A_2 & B \\ 0 & A_0 \end{pmatrix}$$

where A_2 and A_0 are matrices for ϕ_2 and ϕ_0, respectively. It follows that $\mathrm{Tr}(\phi_1) = \mathrm{Tr}(\phi_2) + \mathrm{Tr}(\phi_1)$, as required. The general case now follows by induction. If $k > 3$ and F is the image of F_2 in F_1, we have complexes

$$0 \to F_h \to \cdots \to F_2 \to F \to 0$$

and $0 \to F \to F_1 \to F_0 \to 0$. The endomorphism $\phi_\bullet$ induces endomorphisms of both these complexes: the only new map ψ needed is a map $F \to F$, and this may be taken to be either the restriction of ϕ_1, which stabilizes the kernel of $F_1 \to F_0$, or by ϕ_2, which induces a map of $\mathrm{Coker}\,(F_3 \to F_2)$ to itself. By the cases of complexes whose length is 3 and whose length is $k - 1$, we have that the alternating sum of the traces of the maps is 0 for each of these complexes. When we add, $\mathrm{Tr}(\psi)$ occurs twice, with opposite signs, and the result we want follows. $\square$