Math 711: Lecture of December 7, 2005

We now use the notion of trace developed last time to prove:

Theorem. *Let R be a Noetherian ring containing $\mathbb{Q}$ and let S be a module-finite extension of R such that $\mathrm{pd}_R S < \infty$. Then R is a direct summand of S.*

Proof. The issue is local on R, and so we may assume that R is local. Each element $s \in S$ gives an R-linear endomorphism ϕ_s of S, namely multiplication by s. The injection $R \hookrightarrow S$ shows that if we localize at the multiplicative system W of all nonzerodivisors in S, $W^{-1}S$ is $W^{-1}R$-free of rank ρ at least one. The map $s \mapsto \dfrac{1}{\rho}\mathrm{Tr}(\phi_s)$ gives an R-linear retraction from $S \to R$. $\square$

Remark. Of course the proof shows that the result holds whenever ρ is invertible in R: we do not need to assume that R contains $\mathbb{Q}$.

We next discuss some variant notions of tight closure: we shall use one of these to prove a strong form of a result of Evans and Griffith on ranks of modules of syzygies over a regular local ring.

Given a non-empty family of nonzero ideals $\mathcal{C}$ in a Noetherian ring R of characteristic $p > 0$ with the property

$(*)$ if C, $C' \in \mathcal{C}$ then there exists $C'' \in \mathcal{C}$ such that $C'' \subseteq C \cap C'$

we can define the tight closure with respect to $\mathcal{C}$: an element $u \in N \subseteq M$ is in the *tight closure with respect to* $\mathcal{C}$ of N in M if there exists an ideal $C \in \mathcal{C}$ such that $Cu^q \in N^{[q]}$ for all $q = p^e \gg 0$. We can also define the *small tight closure* of N in M with respect to $\mathcal{C}$: for this we require that for some $C \in \mathcal{C}$, $Cu^q \in N^{[q]}$ for all q (which includes $q = 1$). The property $(*)$ is needed so that the tight closure of N will be closed under addition.

If we take the family $\mathcal{C}$ to consist of all principal ideals generated by an element of R°, we obtain the usual notion of tight closure.

If the family consists of only the unit ideal R, tight closure with respect to this family is Frobenius closure, while the small tight closure of N is the submodule N itself.

If R has a test element, tight closure with respect to the family consisting of the single ideal it generates gives ordinary tight closure, as does small tight closure with respect to the family consisting of the single ideal it generates.

We note that iterating one of these variant tight closure operations may give a larger result than performing it once. One can show that iterating the operation gives the same result if the family of ideals has the property that for all C, $C' \in \mathcal{C}$, there exists $C'' \in \mathcal{C}$ such that $C'' \in CC'$.

We now want to show how one of these variant notions of tight closure can by used to prove the Evans-Griffith syzygy theorem. We want to make two remarks. First, it is immediate from the definition that $u \in M$ is in the tight closure (respectively, small tight closure) with respect to $\mathcal{C}$ of N in M if and only if the image of u in M/N is in the tight closure (respectively, small tight closure) of 0 in M/N with respect to $\mathcal{C}$. The second remark we state as:

Lemma. *If (R, m, K) is local, $\mathcal{C}$ is a non-empty family of nonzero ideals of R, and x is a minimal generator of a finitely generated module M, then x is not in the tight closure (nor in the small tight closure) of 0 in M with respect to $\mathcal{C}$.*

Proof. If u is in the tight closure of 0 in M we have that $Cx^q = 0$ in $F^e(M)$ for all $q \gg 0$. We can map $M \twoheadrightarrow K$ so that $x \mapsto 1$. We get an induced surjection $F^e(M) \to R/m^{[q]}$. It follows that $C \subseteq m^{[q]}$ for all $q \gg 0$, which implies that $C = (0)$, a contradiction. $\square$

We shall need to make use of the notion of order ideal. Let x be an element of M, a finitely generated module over a Noetherian ring R. We define the *order ideal* $\mathfrak{O}_M(x) = \mathfrak{O}(x)$ to be $\{f(x) : f \in \mathrm{Hom}_R(M, R)\}$. For finitely generated modules over a Noetherian ring R, the formation of the order ideal commutes with localization.

The map $R \to M$ sending $1 \mapsto x$ evidently splits if and only if $\mathfrak{O}_M(x) = R$.

Also note that for any finitely generated free R-module G, any R-linear map $M \to G$ takes x into $\mathfrak{O}_M(x)G$.

The Evans-Griffith syzygy theorem asserts that, a $k\,th$ module of syzygies over a regular local ring, if not free, has rank at least k. They prove more general statements, in which the conditions on the ring are weakened but the module is assumed to have finite projective dimension. However, the key point in their proof is the following:

Theorem (Evans-Griffith). *Let R be a local ring containing a field, let M be a $k\,th$ module of syzygies of a finitely generated module of finite projective dimension, and suppose that M_P is R_P-free for every prime P of R except the maximal ideal, i.e., M is locally free on the punctured spectrum of R. Let $x \in M$ be a minimal generator. Then $\mathfrak{O}(x)$ is either the unit ideal or else has height at least k.*

In fact, they show that this is true by using the fact that the improved new intersection theorem is true when R contains a field, which they deduce from the existence of big Cohen-Macaulay modules in the equal characteristic case. We shall eventually give their argument, but we first prove a better result in characteristic p, with depth replacing height and without the assumption that M is locally free on the punctured spectrum. We use a variant notion of tight closure in the argument.

Theorem. *Let (R, m, K) be a local ring of prime characteristic $p > 0$ and let N be a finitely generated module of finite projective dimension over R. Let M be a finitely generated $k\,th$ module of syzygies of N, and let $x \in M$ be a minimal generator of M. Let $I = \mathfrak{O}_M(x)$. Then either $I = R$ or else $\mathrm{depth}_I R \geq k$.*

Proof. If not, let $y_1, \dots, y_d$ be a maximal regular sequence in the proper ideal I, and let $J = (y_1, \dots, y_d)R$. Then we can choose $c \in R - J$ such that $cI \subseteq J$. Let c' denote the image of c in $R' = R/J$. Let $G_\bullet$ be a resolution of N by finitely generated free modules over R such that $G_k \to G_{k-1}$ factors $G_k \twoheadrightarrow M \hookrightarrow G_{k-1}$, which we know exists because M is $k\,$th module of syzygies of N over R. Let B denote the image of G_{k+1} in G_k. Let

G'_j denote $R' \otimes_R G_j$, while M' denotes $R' \otimes_R M$ and B' denotes the image of $R' \otimes_R B$ in G'_k. Choose an element z of G_k that maps onto $x \in M$. We shall obtain a contradiction by showing that the image z' of z in G'_k is in the tight closure of B' in G'_k with respect to the family $\{c'R'\}$. This implies that the image x' of x in M' is in the tight closure of 0 in M' with respect to the family $\{c'R'\}$, a contradiction using the Lemma above, because x' is a minimal generator of M'.

To see this, note that $F^e_R(G_\bullet)$ remains acyclic for all e: the determinantal ranks of the maps and the depths of the ideals of minors do not change. Thus, this is a free resolution of $F^e_R(N)$, and it follows that $R' \otimes_R F^e_R(G_\bullet)$ has homology $\mathrm{Tor}^R_\bullet\big(R', F^e_R(N)\big)$. Since $\mathrm{pd}_R R' = d < k$, we have that $\mathrm{Tor}^R_k\big(R', F^e_R(N)\big) = 0$. But the complex $R' \otimes_R F^e_R(G_\bullet)$ may be identified with $F^e_{R'}(G'_\bullet)$. Let d' denote the map $G'_k \to G'_{k-1}$. Now consider the value of the R-linear map $F^e_{R'}(d')$ evaluated on $c'(z')^q$. This is evidently $c'F^e_{R'}(d')\big((z')^q)\big)$. Since the map $G_k \to G_{k-1}$ factors through M, the image of z, which maps to $x \in M$, is in IG_{k-1}. It follows that the image of z' under d' in IG'_{k-01}, and, hence, that the image of $(z')^q$ under $F^e(d')$ is in

$$I^{[q]}F^e(G'_{k-1}) \subseteq IF^e_{R'}(G'_{k-1}).$$

Since $cI \subseteq J$ and J becomes 0 in R', we have that $F^e_{R'}(d')\big(c'(z')^q\big) = 0$. Since $c'(z')^q$ is a cycle and the homology at this spot is 0, it follows that $c'(z')^q$ is a boundary, which means that it is in the image $(B')^{[q]}$ of $F^e_{R'}(B')$. Thus, z' is in the tight closure with respect to the family $\{c'R'\}$ of B' in G'_k, and this means that x' is in the tight closure with respect to $\{cR'\}$ of 0 in M'. Since x is a minimal generator of M and $J \subseteq m$, it follows that x' is a minimal generator of M', and we have obtained the contradiction of the preceding Lemma mentioned earlier. $\square$