Math 711: Lecture of December 9, 2005

To conclude the proof of the syzygy theorem in characteristic p, we shall make use of the following preliminary result that is independent of the characteristic.

Lemma. *Let M be finitely generated module of finite projective dimension over a Noetherian ring R. Then M is a k th module of syzygies, where $k \in \mathbb{N}$, if and only if for every prime ideal P of R, either M_P is R_P-free or M_P has depth at least k on PR_P.*

Proof. We first show that the condition is necessary. Suppose that M is a k th module of syzygies of N, which will also have finite projective dimension. For any prime P such that M_P is not R_P-free, we have that

$$\mathrm{depth}_{PR_P} M_P = \mathrm{depth}_{PR_P} R_P - \mathrm{pd}_{R_P} M_P = \mathrm{depth}_{PR_P} R_P - (\mathrm{pd}_{R_P} N_P - k)$$

$$= (\mathrm{depth}_{PR_P} R_P - \mathrm{pd}_{R_P} N) + k = \mathrm{depth}_{PR_P} N_P + k \geq k,$$

as required.

To prove the converse, suppose that for every prime P, either M_P is free or M_P has depth at least k. If $k = 0$ there is nothing to prove. Suppose that $k \geq 1$ and choose a basis $f_1, \ldots, f_h$ for $\mathrm{Hom}_R(M, R)$. Then the maps $(f_1, \ldots, f_h)$ give a map $M \to R^h$. We shall show this map is injective and that the cokernel satisfies the condition to be a $(k-1)$ st module of syzygies: this will complete the argument, by induction.

To show injectivity it will suffice to show that given $x \in M - \{0\}$, there is an R-linear map $f : M \to R$ such that $f(x) \neq 0$, for f must be an R-linear combination of the f_j, and it follows that some $f_j(x)$ is nonzero. To prove the existence of f, first replace x by a nonzero multiple whose annihilator is a prime P. Then M_P has depth 0 over R_P. Since $k \geq 1$, it follows that M_P is free over R_P. Hence we can certainly find a map $M_P \to R_P$ over R_P that is nonzero on $x/1$. This map has a multiple by an element of $R - P$ that is the image of a map $f : M \to R$, and $f(x) \neq 0$.

It remains only to show that $C = R^h/M$ is a $(k-1)$ st module of syzygies. Note that C has finite projective dimension. If we localize at any prime P, we have an exact sequence

$$(*) \quad 0 \to M_P \to R_P^h \to C_P \to 0.$$

There are two cases. If M_P is not free, then it has depth at least k. Since M_P has finite projective dimenson over R_P the sum of the depth of M_P and its projective dimension is the depth of R_P. Thus, R_P has depth at least k, and so does R_P^h. The short exact sequence $(*)$ displayed above then implies that the depth of C_P is at least $k-1$. (One may see this from the long exact sequence for $\mathrm{Ext}^{\bullet}_{R_P}(R_P/PR_P, \_)$.)

In the remaining case, where M_P is free over R_P, we claim that the sequence $(*)$ splits, so that C_P is free as well. This will complete the proof. First note that

$$\mathrm{Hom}_R(M, R)_P \cong \mathrm{Hom}_{R_P}(M_P, R_P).$$

It follows that the images of the f_i generate the latter.

We change notation now, and write R instead of R_P. It suffices to prove that when M is R-free and the f_i generate $\mathrm{Hom}_R(M,\,R)$, then the map $\phi : M \to R^h$ given by $u \mapsto (f_1,\,\ldots\,,f_h)$ splits. Let $b_1,\,\ldots\,,b_s$ be a free basis for M. Use this basis to identify M with R^s, and let $g_i : M \to R$ be the map that sends b_i to $1 \in R$ and kills the other b_ν. Then

$$g_i = \sum_{j=1}^{h} r_{i,j} f_j$$

for each i, where $r_{i,j} \in R$ and $(r_{i,j})$ is an $s \times h$ matrix over R. This matrix represents a map $\theta : R^h \to R^s$. This is the required splitting, since

$$\theta\phi(b_i) = \theta\bigl(f_1(b_i),\,\ldots\,,f_h(b_i)\bigr),$$

whose ν th coordinate is

$$\sum_{j=1}^{h} r_{\nu,j} f_j(b_i) = g_\nu(b_i),$$

which is 0 if $\nu \neq i$ and is 1 if $\nu = i$, i.e., $\theta\phi(b_i) = b_i$, as required. □

Proof of the syzygy theorem in characteristic p. We now use the result on the depth of the order ideal and the lemma above to prove that if a k th module of syzygies M over a local ring R has finite projective dimension then either M is free or R^k embeds in M. We use induction on k. Let x be a minimal generator of M. Then $\mathfrak{O}_M(x)$ has depth at least k, or is the unit ideal. It follows that there is a map $M \to R$ whose value on x is not a zerodivisor. This implies that the annihilator of x in R is 0, and so the map $R \to M$ sending $r \mapsto rx$ is injective. We then have a short exact sequence

$$0 \to R \to M \to M/Rx \to 0.$$

M/Rx still has finite projective dimension. If we can show that it is still at least a $(k-1)$ st syzygy, it will follow from the induction hypothesis that we have an injection $R^{k-1} \hookrightarrow M/Rx$, This map lifts to a map $R^{k-1} \to M$ whose image G must be disjoint from Rx, and so $Rx + G \cong Rx \oplus G$ is the required rank k free submodule of M.

To see that M/Rx is a $(k-1)$ st syzygy, consider what happens when we localize at a prime P. We consider two cases. If P contains $\mathfrak{O}_M(x)$, then R_P has depth at least k, and so does M_P (if it is free, that holds because R_P has depth at least k). It follows that M/Rx has depth at least $k-1$. If P does not contain the order ideal, then $R_P x$ splits from M_P. This implies that $(M/Rx)_P$ is free if M_P is, and otherwise has at least the same depth as M_P, and so has depth $\geq k > k-1$. Thus, M/Rx is a k th syzygy, and the argument is complete. □

We next want to give an alternative proof of the syzygy theorem that is very close to the original proof of Evans and Griffith, although what we prove is a bit stronger. Their argument shows that the syzygy theorem follows from the improved new intersection conjecture and, hence, from the direct summand conjecture. However, when R is not a domain, the conclusion is slightly weaker, in that we only show that M is free or else that R_P^k embeds in M_P for some prime P of R. This is the same as asserting that M has torsion-free rank k when R is a domain.

We shall proceed by assuming that M is a counterexample and then localizing as much as possible so that M remains not free. In this way, we can reduce to the case where M is locally free on the punctured spectrum of R. We shall then be able to apply the improved new intersection theorem to the complex obtained by apply $R/\mathfrak{O}_M(x) \otimes_R \_$ to a free resolution of M over R.