Math 711: Lecture of December 12, 2005

We now give an alternative proof of the syzygy conjecture, closer in spirit to the original paper of Evans and Griffith. We first give a new proof, in the special case where the module of syzygies is locally free on the punctured spectrum, of the result on the depth of the order ideal. We need a preliminary result.

Lemma. *Let M be a finitely generated module over a local ring $(R,\, m,\, K)$ that is locally free on the punctured spectrum of R. Let x be an element of M, and let $I = \mathfrak{O}_M(x)$. Then the image of x is M/IM is killed by a power of m.*

Proof. Consider the set

$$W = \{u \in R : M_u \text{ is } R_u\text{-free}\}.$$

Then W must generate m, or else it is contained in a prime ideal P strictly smaller than M. But, since M_P is R_P-free, this is also true for some $u \in R - P$, a contradiction. Let Suppose $u_1, \, \ldots \,, u_h \in W$ generate an ideal primary to m. Then it suffices to show that for every u_j there is an integer n_j such that $u_j^{n_j} x \in IM$. For then the image of x in M/IM is killed by the m-primary ideal generated by the $u_j^{n_j}$, $1 \le j \le h$. Thus, it suffices to show that if M_u is free, then $u^n x \in IM$ for some n. Consider mutually inverse isomorphisms $M_u \cong R_u^k \cong M_u$. After multiplying by some power of u, we may assume that these maps are induced by maps $f : M \to R^k$ and $g : R^k \to M$ such that $g \circ f : M \to M$ has the property that $g \circ f - u^n \mathrm{id}_M$ induces the 0 map on M. After multiplying further by a power of u, we may assume $g \circ f = u^n \mathrm{id}_M$ (n may have increased). By the definition of $\mathfrak{O}_M(x) = I$, $f(x) \in IR^k$, and so $(g \circ f)(x) \in IM$. But this says that $u^n x \in IM$, as required. $\square$

Theorem. *Let $(R,\, m,\, K)$ be local, let M be a finitely generated R-module that is a k th module of syzygies of finite projective dimension over R, let x be a minimal generator of M, and let $I = \mathfrak{O}_x(M)$. If M is locally free on the punctured spectrum of R, and the improved new intersection theorem holds for R/I, then $I = R$ or else* $\mathrm{depth}_I R = k$.

Proof. Let M be a k th module of syzygies of N, and let

$$0 \to G_h \to \cdots \to G_0 \to 0$$

denote a minimal free resolution of M. Since M is not free, we must have $\mathrm{pd}_R M = \mathrm{pd}_R N - k$. Because M is locally free on the punctured spectrum,

$$\mathrm{Tor}_j^R(M,\, R/I)_u \cong \mathrm{Tor}_j^{R_u}(M_u,\, R_u/IR_u) = 0$$

for $j \ge 1$, and it follows that $\mathrm{Tor}_j^R(M,\, R/I)$ has finite length for $j \ge 1$. We may assume $I \ne R$. Let $S = R/I$. Then the higher homology of the S-free complex $S \otimes_R G_\bullet$ has finite length. By the preceding Lemma, the image of x in $H_0(S \otimes_R G_\bullet) \cong M/IM$ is killed by a power of the maximal ideal, and it is a minimal generator of M/IM. Therefore, we may apply the improved new intersection theorem to conclude that

$$\dim(R/I) \le h = \mathrm{pd}_R N - k \le \mathrm{depth}_m R - k.$$

Then $\mathrm{depth}(R) - \dim(R/I) \ge k$. By the first Lemma of the Lecture of November 21 with $M = R/I$, we have

$$\mathrm{depth}(R) \le \mathrm{depth}_I R + \dim(R/I),$$

i.e.,

$$\mathrm{depth}_I R \ge \mathrm{depth}(R) - \dim(R/I) \ge k,$$

as required. $\square$

The following result then recovers the syzygy theorem when R is a domain: in particular, it recovers the syzygy theorem when R is regular, which is the most important case.

Theorem. *Let M be a k th module of syzygies of a module of finite projective dimension over a local ring R such that the improved new intersection theorem holds for homomorphic images of localizations of R. Then either M is free, or M_P contains R_P^k as a submodule for some prime P of R.*

Proof. Choose a counterexample with $\dim(R)$ minimum and k minimum. We may assume that M is not free. We may also assume that M is locally free on the punctured spectrum, or else we can localize to obtain a counterexample with a ring of smaller dimension. Therefore, the Theorem above applies, and for a minimal generator $x \in M$, either $\mathfrak{O}_M(x) = R$ or else it has depth at least k. We may assume that $k \ge 1$ or there is nothing to prove. We can conclude that x has annihilator 0, hence that $R \cong Rx$ and that M/xR is a $(k-1)$ st module of syzygies exactly as in the proof of the syzygy theorem in characteristic p given on the second page of the Lecture Notes of December 9. We may apply the induction hypothesis to M/xR. We can conclude that (M_P/xR_P) has a an R_P-free submodule of rank $k-1$, and the argument that M_P has an R_P-free submodule of rank k is likewise the same as in the proof from the Lecture Notes of December 9. $\square$

The syzygy theorem is quite non-trivial even in regular rings and for small values of k. Here is a remarkable consequence:

Theorem (Evans-Griffith). *If I is an unmixed height two ideal with three generators in a regular local ring of dimension at least 3 containing a field, then R/I is Cohen-Macaulay.*

Proof. Consider a minimal resolution

$$0 \to M \to R^3 \to I \to 0.$$

Then M has rank 2. We want to show that M is free. To do so, it suffices to prove that it is a third syzygy, for then, if it is not free, it cannot have rank 2. Note that M is a second syzygy, since I is a first syzygy of R/I. To show that M is a third syzygy, we need only show that for every prime ideal P, either M_P is free or has depth at least 3. If P has height at most two, it follows from the fact that M_P is a second syzygy over R_P that it is free. Therefore, we may assume that P has height 3 or more. If P does not contain I, then $IR_P = R_P$, so that the sequence

$$0 \to M_P \to R_P^3 \to IR_P \to 0$$

splits, and M_P is free. Therefore, we may assume that P contains I. Then, since I is unmixed of height two and $\dim(R_P) \geq 3$, we have that $\mathrm{depth}_{PR_P} R_P/IR_P \geq 1$. But then the depth of IR_P on PR_P is at least two, and the depth of M_P on PR_P is at least three, as required.

Thus, M is free, and $\mathrm{pd}_R I = 1$, so that $\mathrm{pd}_R(R/I) = 2$. Then

$$\mathrm{depth}_m(R/I) = \dim(R) - 2 = \dim(R/I),$$

and R/I is Cohen-Macaulay, as required. $\square$

We next want to mention a problem on which there has been little progress. Serre proved that if M and N are nonzero modules over a regular local ring R and $M \otimes_R N$ has finite length, then $\dim(M) + \dim(N) \leq \dim(R)$. By the philosophy of the "metaconjecture" that results over regular rings should generalize to the case of finite projective dimension, it is reasonable to conjecture that the same holds without the condition that R be regular if $\mathrm{pd}_R M$ is finite. This is an open question.

Similarly, Serre defined the intersection multiplicity $\chi(M, \mathbb{N})$ of a pair of finitely generated R-modules M, N when $\ell(M \otimes_R N) < \infty$ over a regular local ring (R, m, K) of Krull dimension d as

$$\sum_{j=0}^{d} (-1)^j \ell\big(\mathrm{Tor}_j^R(M, N)\big).$$

One can define χ as well when R is local and $\mathrm{pd}_R M$ is finite.

It is known in the regular case that if $\dim(M) + \dim(N)\dim(R)$ then this multiplicity vanishes and that if $\dim(M) + \dim(N) = \dim(R)$ then it is nonnegative, and it is conjectured that it is actually positive. The conjecture is known if $\widehat{R}$ is a formal power series over a field or a DVR (hence, the conjecture is known if R contains a field), or if $\dim(R) \leq 4$. The general case remains open.

The metaconjecture philosophy suggests the conjecture that if M, N are finitely generated modules over a local ring such that $\ell(M \otimes_R N) < \infty$ then $\chi(M, N) = 0$.

While this conjecture is still open if both M and N have finite projective dimension (and is known in that case if the ring is a complete intersection: see [P. Roberts, *Multiplicities and Chern classes in local algebra*, Cambridge Tracts in Mathematics **133**, Cambridge Univ. Press, Cambridge, UK, 1998], §13, and [P. Roberts, *The vanishing of the intersection multiplicities of perfect complexes*, Bull. A.M.S. **13** (1985) 127–130]), the conjecture is known to be false in general, even when the ring is a hypersurface. In [S. Dutta, M. Hochster, J. E, McLaughlin, *Modules of finite projective dimension with negative intersection multiplicities*, Invent. Math. **79** (1985) 253–291] a counterexample is given, which we shall discuss below.

Let $R = K[u,\, v,\, x,\, y]_m/(uy - vx)$, where K is any field and $m = (u,\, v\, x,\, y)$. We shall construct a module M of finite length and projective dimension 3 such that, with $P = (u,\, v)R$, we have that $\chi(M,\, R/P) = -1$. Note that $\dim(M) + \dim(R/P) = 0 + 2 < 3 = \dim(R)$.

The module M is contructed as a finite-dimensional K-vector space, with the actions of u, v, x, and y given by four specific commuting nilpotent square matrices A, B, C, and D, respectively, such that $AD = BC$. We first need an effective criterion for when a module has finite projective dimension. The following fact will be useful.

Lemma. *Let M be a finitely generated module over $R = K[u,\, v\, x,\, y]_m/(uy - vx)$ as above, and $P = (u,\, v)R$. Then $\mathrm{pd}_R M$ is finite if and only if for all $j \gg 0$, $\mathrm{Tor}_j^R(M,\, R/P) = 0$.*

Proof. If one has a short exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

and one knows that $\mathrm{Tor}_j^R(M,\, N_i) = 0$ for $j \gg 0$ for two values of i in $\{1,\, 2,\, 3\}$, then it follows from the long exact sequence for Tor that $\mathrm{Tor}_j^R(M,\, N_i) = 0$ for $j \gg 0$ for all three values of i. It follows by induction on the length of the sequence that if one has an exact sequence

$$0 \to N_1 \to \cdots \to N_h \to 0$$

such that $\mathrm{Tor}_j^R(M,\, N_i) = 0$ for all $j \gg 0$ and all but one value i_0 of i in $\{1,\, 2,\, \ldots\,,\, h\}$, then $\mathrm{Tor}_j(M,\, N_i) = 0$ for all $j \gg 0$ and all $i \in \{1,\, 2,\, \ldots\,,\, h\}$. Let N be the image of N_{h-2} in N_{h-1} and consider instead the two shorter sequences

$$0 \to N_1 \to \cdots \to N_{h-2} \to N \to 0$$

and

$$0 \to N \to N_{h-1} \to N_h \to 0.$$

One of the two (the one in which N_{i_0} does not occur) can be used to show that $\mathrm{Tor}_j^R(M,\, N) = 0$ for all $j \gg 0$, and the other to show the same condition holds for N_{i_0},

Now, $R/P = K[x,\, y]_{(x,y)}$, and the augmented Koszul complex of x, y over this ring gives an exact sequence

$$0 \to R/P \to R/P \oplus R/P \to R/P \to K \to 0.$$

We may simply think of this sequence as an exact sequence of R-modules. The remarks of the preceding paragraph show that $\mathrm{Tor}_j(M,\, K) = 0$ for all $j \gg 0$. Since the K-vector space dimensions of these Tor modules give the ranks of the free modules in a minimal free resolution of M over R, it follows that $\mathrm{pd}_R M < \infty$. $\square$

We next want to see what the minimal free resolution of R/P looks like explicitly. Like any minimal free resolution over a hypersurface, we know that it will be eventually periodic of period at most 2.