Math 711, Fall 2005 **Problem Set #2**
Due: Friday, October, 28

1. Let $R$ be a finitely generated $\mathbb{N}$-graded algebra such that $R_0 = K$, a field. Let $m$ be the unique homogeneous maximal ideal of $R$. Prove that $R_m$ is a domain (respectively, reduced) if and only if $R$ is a domain (respectively, reduced).

2. Let $K$ be a field, and let $X$ be an $r \times s$ matrix over $K$, where $r \leq s$. Let $I = I_r(X)$, the ideal generated by the maximal size minors of $X$. Consider the $s - r + 1$ diagonals of $X$ that begin with $x_{1,j}$, where $1 \leq j \leq s - r + 1$, and whose other elements are $x_{1+k,j+k}$, $1 \leq k \leq r - 1$. Consider the indeterminates that are not on any of these diagonals (there are $1+2+\cdots+r-1$ of these at the left, below the diagonal for $j = 1$, and the same number on the right, above the diagonal for $j = s - r + 1$), and the $(s - r + 1)(r - 1)$ differences of consecutive elements on the same diagonal, each of which has the form $x_{1+k,i+k} - x_{k,i+k1}$, where $i$ and $j$ are constrained as before. Show that the images of these elements form a homogeneous system of parameters for $R/I_r(X)$.

3. In the problem above, show that if one kills this homogeneous system of parameters in $K[X]/I_r(X)$, the ring obtained is isomorphic with $K[x_{1,1}, \ldots, x_{1,s-r+1}]/m^r$, where $m$ is the ideal generated by the $x_{1,j}$. Determine the type of $K[X]/I_r(X)$ when $X$ is an $r \times s$ matrix of indeterminates.

4. Let $K$ be an algebraically closed field of positive characteristic $p > 0$, where $p \neq 3$. Let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$. Show that $R$ is F-injective if and only if $p \equiv 2 \bmod 3$.

5. Let $R$ be a finitely generated algebra over a field of characteristic $p > 0$. Is the set of primes $P$ such that $R_P$ is an F-injective Cohen-Macaulay local ring Zariski open?