Math 711, Fall 2005 **Problem Set #5**
Due: Thursday, December 22

1. Let R be Noetherian and let G be a finitely generated module that is locally free. Suppose that $\mathrm{Spec}\,(R)$ is connected. Prove that the rank of G_P over R_P does not depend on the prime P.

2. Let R be a semilocal Noetherian ring and G a finitely generated module such that for every maximal ideal m of R, G_m is free of rank r. Prove that G is free of rank r.

(The results of 1. and 2. were assumed in the discussion of the trace of an endomorphism of a module of finite projective dimension.)

3. Let R be a Cohen-Macaulay local ring of dimension d that is a homomorphic image of a complete regular local ring of dimension n. Let E be the injective hull of the residue class field. Let $\omega = \mathrm{Ext}_S^{n-d}(R, S)$, which is a finitely generated R-module. Consider the full subcategory $\mathcal{C}$ of R-modules whose objects are the finitely generated R-modules of finite injective dimension. Are the functors $\mathrm{Hom}_R(\omega, \_)$ and $\mathrm{Ext}_R^d(E, \_)$ isomorphic on $\mathcal{C}$?

4. Let c be a nonzero element of R. Let $I^{\#}$ denote the tight closure of the ideal I with respect to the family $\{cR\}$. Show that it is possible that $(I^{\#})^{\#}$ contains $I^{\#}$ strictly.

5. Show that the canonical element of a local ring (R, m, K) of Krull dimenson d may be viewed as an element of $\mathrm{Tor}_d^R\big(H_m^d(R), K\big)$.