Math 711: Lecture of September 8, 2006

Corollary. *If R is integrally closed in S, then $R[t]$ is integrally closed in $S[t]$. If R is normal, then $R[t]$ is normal.*

Proof. The integral closure of $R[t]$ in $S[t]$ will be graded and so spanned by integral elements of $S[t]$ of the form st^k, where s is homogenous. Take an equation of integral dependence for st^k of degree, say, n on $R[t]$. The coefficient of t^{kn} is 0, and this gives an equation of integral dependence for s on R. For the second part, $R[t]$ is integrallly closed in $K[t]$, where $K = \mathrm{frac}\,(R)$, and $K[t]$ is integrally closed in $K(t) = \mathrm{frac}\,(R[t])$ since $K[t]$ is a UFD. $\square$

We next want to discuss integral closure of ideals.

Let R be any ring and let I be an ideal of R. We define an element u of R to be *integral over* I or to be in the integral closure $\overline{I}$ of I if it satisfies a monic polynomial $f(z)$ of degree n with the property that the coefficient of z^{n-t} is in I^t, $1 \leq t \leq n$. We shall use the temporary terminology that such a monic polynomial is *I-special*. Note that the product of two I-special polynomials is I-special, and hence any power of an I-special polynomial is I-special.

Let t be an indeterminate over R and let $R[It]$ denote the subring of the polynomial ring $R[t]$ generated by the elements it for $i \in I$. This ring is called the *Rees ring* of I. It is $\mathbb{N}$-graded, with the grading inherited from $R[t]$, so that the k th graded piece is It^k.

It follows easily that the integral closure of $R[It]$ in $R[t]$ has the form

$$R + J_1 t + J_2 t^2 + \cdots + J_k t^k + \cdots$$

where, since this is an R-algebra, each J_k is an ideal of R. We note that with this notation, $J_1 = \overline{I}$. To see this, note that if rt, where $r \in R$ is integral over $R[It]$, satisfying an equation of degree n, then by taking homogeneous components of the various terms we may find an equation of integral dependence in which all terms are homogeneous of degree n. Dividing though by t^n then yields an equation of integral dependence for r on I. This argument is reversible. (Exercise: in this situation, show that J_k is the integral closure of I^k.) We note several basic facts about integral closures of ideals that follow easily either from the definition or this discussion.

Proposition. *Let I be an ideal of R and let $u \in R$.*

(a) *The integral closure of I in R is an ideal containing I, and the integral closure of $\overline{I}$ is $\overline{I}$.*

(b) *If $h : R \to S$ is a ring homomorphism and u is integral over I then $h(u)$ is integral over IS. If J is an integrally closed ideal of S then the contraction of J to R is integrally closed.*

(c) *u is integral over I if and only if its image modulo the ideal N of nilpotent elements is integral over $I(R/N)$. In particular, the integral closure of (0) is N.*

(d) *The element u is integral over I if and only if for every minimal prime P of R, the image of u modulo P is integral over $I(R/P)$.*

(e) *Every prime ideal of R and, more generally, every radical ideal of R is integrally closed.*

(f) *An intersection of integrally closed ideals is integrally closed.*

(g) *In a normal domain, a principal ideal is integrally closed.*

(h) *If S is an integral extension of R then $\overline{IS} \cap R = \overline{I}$.*

Proof. That $I \subseteq \overline{I}$ is obvious. If r is in the integral closure of $\overline{I}$ then rt is integral over $R[\overline{I}t]$. But this ring is generated over $R[It]$ by the elements $r_1 t$ such that $r_1 \in \overline{I}$, i.e., such that $r_1 t$ is integral over $R[It]$. It follows that $R[\overline{I}t]$ is integral over $R[It]$, and then rt is integral over $R[It]$ by the transitivity of integral dependence. This proves (a).

The first statement in (b) is immediate from the definition of integral dependence: apply the ring homomorphism to the equation of integral dependence. The second statement in (b) is essentially the contrapositive of the first statement.

The "only if" part of (c) follows from (b) applied with $S = R/N$. The "if" part follows from (d), and so it will suffice to prove (d).

The "only if" part of (d) likewise follows from (b). To prove the "if" part note that the values of I-special polynomials on u form a multiplicative system: hence, if none of them vanishes, we can choose a minimal prime P of R disjoint from this multiplicative system, and then no $I(R/P)$-special polynomial vanishes on the image of u in R/P.

(e) follows from the second statement in (c) coupled with the second statement in (b), while (f) is immediate from the definition. To prove (g), suppose that b is an element of R and $a \in \overline{bR}$. If $b = 0$ it follows that $a = 0$ and we are done. Otherwise, we may divide a degree n equation of integral dependence for a on bR by b^n to obtain an equation of integral dependence for a/b on R. Since R is normal, this equation shows that $a/b \in R$, and, hence, that $a \in bR$.

Finally, suppose that $r \in R$ is integral over IS. Then rt is integral over $S[ISt]$, and this ring is generated over $R[It]$ by the elements of S, each of which is integral over R. It follows that $S[ISt]$ is integral over $R[It]$, and so rt is integral over $R[It]$ by the transitivity of integral dependence. $\square$

Recall that a domain V with a unique maximal ideal m is called a *valuation domain* if for any two elements one divides the other. This implies that for any finite set of elements, one of the elements divides the others, and so generates the same ideal that they all do together. We shall use the term *discrete valuation ring*, abbreviated DVR, for a Noetherian valuation domain: in such a ring, the maximal ideal is principal, and every nonzero element of the maximal ideal is a unit times a power of the generator of the maximal ideal. A DVR is the same thing as a local principal ideal domain (PID).

We recall the following terminology: (R,m,K) is *quasilocal* means that R has unique maximal ideal m and residue class field $K = R/m$. Sometimes K is omitted from the notation. We say that (R, m, K) is *local* if it is quasilocal and Noetherian.

The next result reviews some facts about integral closures of rings and integrally closed rings.

Theorem. *Let R be an integral domain.*

(a) *R is normal if and only if R is an intersection of valuation domains with the same fraction field as R. If R is Noetherian, these may be taken to be the discrete valuation rings obtained by localizing R at a height one prime.*

(b) *If R is one-dimensional and local, then R is integrally closed if and only if R is a DVR. Thus, a local ring of a normal Noetherian domain at a height one prime is a DVR.*

(c) *If R is Noetherian and normal, then principal ideals are unmixed, i.e., if $r \in R$ is not zero not a unit, then every associated prime of rR has height one.*

Proof. For (a) and (b) see [M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1969], Corollary 5.22, Proposition 9.2 and Proposition 5.19, respectively, and [O. Zariski and P. Samuel, *Commutative Algebra*, D. Van Nostrand Company, Princeton, New Jersey, 1960], Corollary to Theorem 8, p. 17; for (c) see H. Matsumura, *Commutative Algebra*, W.A. Benjamin, New York, 1970], §17. $\square$

For the convenience of the reader we shall give a proof of (a) below in the case where R is not necessarily Noetherian.