Math 711: Lecture of September 11, 2006

Examples of integral closure of ideals. Note that whenever $r \in R$ and $I \subseteq R$ is an ideal such that $r^n = i_n \in I^n$, we have that $r \in \overline{I}$. The point is that r is a root of $z^n - i_n = 0$, and this polynomial is monic with the required form.

In particular, if x, y are any elements of R, then $xy \in \overline{(x^2,\, y^2)}$, since $(xy)^2 = (x^2)(y^2) \in I^2$. This holds even when x and y are indeterminates.

More generally, if $x_1, \ldots, x_n \in R$ are any elements and $I = (x_1^n, \ldots, x_k^n)R$, then every monomial $r = x_1^{i_1} \cdots x_k^{i_k}$ of degree n (here the i_j are nonnegative integers whose some is n) is in $\overline{I}$, since

$$r^n = (x_1^n)^{i_1} \cdots (x_k^n)^{i_k} \in I^n,$$

since every $x_j^n \in I$ and $\sum_{j=1}^k i_j = n$.

Now let K be any field of characteristic $\neq 3$, and let X, Y, Z be indeterminates over K. Let

$$R = K[X,\, Y,\, Z]/(X^3 + Y^3 + Z^3) = K[x,\, y,\, z],$$

which is a normal domain with an isolated singularity. Here, we are using lower case letters to denote the images of corresponding upper case letters after taking a quotient: we shall frequently do this without explanatory comment. Let $I = (x, y)R$. Then $z^3 \in I^3$, and so $z \in \overline{I}$. This shows that an ideal generated by a system of parameters in a local ring need not be integrally closed, even if the elements are part of a minimal set of generators of the maximal ideal. It also follows that $z^2 \in \overline{I^2}$, where I is a two generator ideal, while $z^2 \notin I$. Thus, the Briançon-Skoda theorem, as we stated it for regular rings, is not true for R. (There is a version of the theorem that *is* true: it asserts that for an n-generator ideal I, $\overline{I^n} \subseteq I^*$, where I^* is the *tight closure* of I. But we are not assuming familiarity with tight closure here.)

We next want to give a proof that, even when a normal domain R is not Noetherian, it is an intersection of valuation domains. We first show:

Lemma. *Let L be a field, $R \subseteq L$ a domain, and $I \subset R$ a proper ideal of R. Let $x \in L - \{0\}$. Then either $IR[x]$ is a proper ideal of $R[x]$ or $IR[1/x]$ is a proper ideal of $R[1/x]$.*

Proof. We may replace R by its localization at a maximal ideal containing I, which only makes the problem harder. Assume that neither is a proper ideal. Since $1 \in IR[x]$ we obtain an equation

$$(\#) \quad 1 = i_0 + i_1 x + \cdots + i_n x^n,$$

where all of the $i_h \in I$. Similarly, we obtain an equation

$$(\#\#) \quad 1 = j_0 + j_1(1/x) + \cdots j_0 m(1/x^m),$$

where all of the $j_h \in I$. We may assume that n and m have been chosen as small as possible. By reversing the roles of x and $1/x$, if necessary, we may assume that $n \geq m$. Then

$$1 - j_0 = j_1(1/x) + \cdots + j_m(1/x)^m.$$

Multiplying by the inverse of $1 - j_0$, we have that

$$1 = j_1'(1/x) + \cdots j_m'(1/x)^m,$$

where the $j_h' \in I$. Multiplying through by x^m yields that

$$x^m = j_1' x^{m-1} + \cdots + j_m' \in I + Ix + \cdots + Ix^{m-1}.$$

It follows by induction on k that for all $k \geq 0$,

$$x^k \in I + Ix + \cdots + Ix^{m-1}.$$

For the inductive step, once we have that

$$x^{k-1} \in I + Ix + \cdots + Ix^{m-1},$$

we can multiply by x to get that

$$x^k \in I + Ix + Ix^2 + \cdots Ix^m,$$

and we can use the fact that

$$x^m \in I + Ix + \cdots + Ix^{m-1}$$

to eliminate the rightmost term on the right. But then we can get rid of the $x^m, \ldots, x^n$ terms in the displayed equation (#), and we have that

$$1 \in I + Ix + \cdots + Ix^{m-1},$$

contradicting the minimality of our choice of n. $\square$

Corollary. *Let $R \subseteq L$, a field, and let $I \subset R$ be a proper ideal of R. Then there is a valuation domain V with $R \subseteq V \subseteq L$ such that $IV \neq V$.*

Proof. Consider the set $\mathcal{S}$ of all rings S such that $R \subseteq S \subseteq L$ and $IS \neq S$. This set contains R, and so is not empty. The union of a chain of rings in $\mathcal{S}$ is easily seen to be in $\mathcal{S}$. Hence, by Zorn's lemma, $\mathcal{S}$ has a maximal element V. We claim that V is a valuation domain with fraction field L. For let $x \in L - \{0\}$. By the preceding Lemma, either $IV[x]$ or $IV[1/x]$ is a proper ideal. Thus, either $V[x] \in \mathcal{S}$ or $V[1/x] \in \mathcal{S}$. By the maximality of V, either $x \in V$ or $1/x \in V$. $\square$

We now can prove the result we were aiming for.

Corollary. *Let R be a normal domain with fraction field L. Then R is the intersection of all valuation domains V with $R \subseteq VincL$.*

Proof. Let $x \in L - R$. It suffices to find V with $R \subseteq V \subseteq L$ such that $x \notin V$. Let $y = 1/x$. We claim that y is not a unit in $R[y]$, for its inverse is x, and if y were a unit we would have

$$x = r_0 + r_1(1/x) + \cdots + r_n(1/x)^n$$

for some positive integer n and $r_j \in R$. Multiplying through by x^n gives an equation of integral dependence for x on R, and since R is normal this yields $x \in R$, a contradiction. Since $yR[y]$ is a proper ideal, by the preceding Corollary we can choose a valuation domain V with $R[y] \subseteq V \subseteq K$ such that yV is a proper ideal of V. But this implies that $x \notin V$. $\square$

The following important result can be found in most introductory texts on commutative algebra, including [M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1969], which we refer to briefly as Atiyah-Macdonald.

Theorem. *If R is a normal Noetherian domain, then the integral closure S of R in a finite separable extension $\mathcal{G}$ of its fraction field $\mathcal{F}$ is module-finite over R.*

Proof. See Proposition 5.19 of Atiyah-MacDonald for a detailed argument. We do mention the basic idea: choose elements $s_1, \ldots, s_d$ of S that are basis for $\mathcal{G}$ over $\mathcal{F}$, and then the discriminant $D = \det\bigl(\mathrm{Trace}_{\mathcal{G}/\mathcal{F}}\, s_i s_j\bigr)$, which is nonzero because of the separability hypothesis, multiplies S into the Noetherian R-module $\sum_{i=1}^{d} Rs_i$. $\square$

Theorem (Nagata). *Let R be a complete local domain. Then the integral closure of R in a finite field extension of its fraction field is a finitely generated R-module.*

Proof. Because R is module-finite over a formal power series ring over a field, or, if R does not contain a field, over a DVR whose fraction field has characteristic zero, we may replace the original R by a formal power series ring, which is regular and, hence, normal. Unless R has characteristic p the extension is separable and we may apply the Theorem just above.

Thus, we may assume that R is a formal power series ring $K[[y_1, \ldots, y_n]]$ over a field K of characteristic p. If we prove the result for a larger finite field extension, we are done, because the original integral closure will be an R-submodule of a Noetherian R-module. This enables us to view the field extension as a purely inseparable extension followed by a separable extension. The separable part may be handled using the Theorem just above. It follows that we may assume that the field extension is contained in the fraction field of $K^{1/q}[[x_1, \ldots, x_n]]$ with $x_i = y_i^{1/q}$ for all i. We may adjoin the x_i to the given field extension, and it suffices to show that the integral closure is module-finite over $K[[x_1, \ldots, x_n]]$, since this ring is module-finite over $K[[y_1, \ldots, y_n]]$. Thus, we have reduced to the case where $R = K[[x_1, \ldots, x_n]]$ and the integral closure S will lie inside $K^{1/q}[[x_1, \ldots, x_n]]$, since this ring is regular and, hence, normal.

Now consider the set $\mathcal{L}$ of leading forms of the elements of S, viewed in the ring $K^{1/q}[[x_1, \dots, x_n]]$. Let d be the degree of the field extension from the fraction field of R to that of S. We claim that any $d+1$ or more $F_1, \dots, F_N$ of the leading forms in $\mathcal{L}$ are linearly dependent over (the fraction field of) R for, if not, choose elements s_j of S which have them as leading forms, and note that these will also be linearly independent over R, a contradiction (if a non-trivial R-linear combination of them were zero, say $\sum_j r_j s_j = 0$, where the r_j are in R, and if F_j has degree d_j while the leading form g_j of r_j has degree d'_j, then one also gets $\sum_j g_j F_j = 0$, where the sum is extended over those values of j for which $d_j + d'_j$ is minimum). Choose a maximal set of linearly independent elements f_j of $\mathcal{L}$. Let K' denote the extension of K generated by all of their coefficients. Since there are only finitely many, $T = K'[[x_1, \dots, x_n]]$ is module-finite over R. But T contains every element L of $\mathcal{L}$, for each element of $\mathcal{L}$ is linearly dependent over R on the f_j, and so is in the fraction field of T, and has its q th power in $R \subseteq T$. Since T is regular, it is normal, and so must contain L.

Thus, the elements of $\mathcal{L}$ span a finitely generated R-submodule of T, and so we can choose a finite set $L_1, \dots, L_k \subseteq \mathcal{L}$ that span an R-module containing all of $\mathcal{L}$,. We can then choose finitely many elements $s_1, \dots, s_k$ of S whose leading forms are the $L_1, \dots, L_k$.

Let S_0 be the module-finite extension of R generated by the elements $s_1, \dots, s_k$. We complete the proof by showing that $S_0 = S$. We first note that for every element L of $\mathcal{L}$, S_0 contains an element s whose leading form is L. To see this, observe that if we write L as an R-linear combination $\sum_j r_j L_j$, the same formula holds when every r_j is replaced by its homogeneous component of degree $\deg L - \deg L_j$. Thus, the r_j may be assumed to be homogeneous of the specified degrees. But then $\sum_j r_j s_j$ has L as its leading form.

Let $s \in S$ be given. Recursively choose $u_0, u_1, \dots, u_n, \dots \in S_0$ such that u_0 has the same leading form as s and, for all n, u_{n+1} has the same leading form a $s - (u_0 + \cdots + u_n)$. For all $n \geq 0$, let $v_n = u_0 + \cdots + u_n$. Then $\{v_n\}_n$ is a Cauchy sequence in S_0 that converges to s in the topology given by the powers m_T^h of the maximal ideal of $T = K'[[x_1, \dots, x_n]]$. Since S_0 is module-finite over $K[[x_1, \dots, x_n]]$, S_0 is complete. By Chevalley's lemma, which is discussed below, when we intersect the m_T^h with S_0 we obtain a sequence of ideals cofinal with the powers of the maximal ideal of S_0. Thus, the sequence, which converges to s, is Cauchy with respect to the powers of the maximal ideal of S_0. Since, as observed above, S_0 is complete, we have that $s \in S_0$, as required. $\square$