Math 711: Lecture of September 15, 2006

The next Theorem gives several enlightening characterizations of integral closure. We first note:

Lemma. . *Let I be an ideal of the ring R, $r \in \overline{I}$, and $h : R \to S$ a homomorphism to a normal domain S such that IS is principal. Then $h(r) \in IS$.*

Proof. By persistence of integral closure, $h(r) \in \overline{IS}$. But IS is a principal ideal of a normal domain, and so integrally closed, which implies that $r \in IS$. $\square$

Theorem. *Let R be a ring, let I be an ideal of R, and let $r \in R$.*

(a) *$r \in \overline{I}$ if and only if for every homomorphism from R to a valuation domain V, $r \in IV$.*

(b) *$r \in \overline{I}$ if and only if for every homomorphism f from R to a valuation domain V such that the kernel of f is a minimal prime P of R, $f(r) \in IV$. (Thus, if R is a domain, $r \in \overline{I}$ if and only if for every valuation domain V containing R, $r \in IV$.) Moreover, it suffices to consider valuation domains contained in the fraction field of R/P.*

(c) *If R is Noetherian, $r \in \overline{I}$ if and only if for every homomorphism from R to a DVR V, $r \in IV$.*

(d) *If R is Noetherian, $r \in \overline{I}$ if and only if for every homomorphism f from R to a DVR V such that the kernel of f is a minimal prime of R, $f(r) \in IV$. (Thus, if R is a domain, $r \in \overline{I}$ if and only if for every DVR V containing R, $r \in IV$.) Moreover, it suffices to consider valuation domains contained in the fraction field of R/P.*

(e) *If R is a domain and $I = (u_1, \ldots, u_n)R$ is a finitely generated ideal, let*

$$S_i = R[\frac{u_1}{u_i}, \ldots, \frac{u_n}{u_i}] \subseteq R_{u_i},$$

and let T_i be the integral closure of S_i. Then $r \in R$ is in $\overline{I}$ if and only if $r \in IT_i$ for all i.

(f) *Let R be a Noetherian domain. Then $r \in \overline{I}$ if and only if there is a nonzero element $c \in R$ such that $cr^n \in I^n$ for all $n \in \mathbb{N}$. (Note: I^0 is to be interpreted as R even if $I = (0)$.)*

(g) *Let R be a Noetherian domain. Then $r \in \overline{I}$ if and only if there is a nonzero element $c \in R$ such that $cr^n \in I^n$ for infinitely many values of $n \in \mathbb{N}$.*

Proof. We first observe that in any valuation domain, every ideal is integrally closed: every ideal is the directed union of the finitely generated ideals it contains, and a directed union of integrally closed ideals is integrally closed. In a valuation domain every finitely generated is principal, hence integrally closed, since a valuation domain is normal, and it follows that every ideal is integrally closed.

Now suppose that $r \in \overline{I}$. Then for any map of f of R to a valuation domain V, we have that $r \in \overline{IV} = IV$. This shows the "only if" part of (a). To complete the proof of both (a) and (b) it will suffice to show that if $f(r) \in IV$ whenever the kernel of f is a minimal prime, then $r \in \overline{I}$. But if $r \notin \overline{I}$ this remains true modulo some minimal prime by part (d) of the Proposition whose statement begins on the first page of the Lecture Notes for September 8, and so we may assume that R is a domain, and that rt is not integral over $R[It]$. But then, by the Corollary at the top of the third page of the Lecture Notes of September 11 , we can find a valuation domain V containing $R[It]$ and not rt (in the Noetherian case V may be taken to be a DVR by the second Theorem on the first page of the Lecture Notes from September 13). Then $r \in \sum_{j=1}^{n} i_j v_j$ with the $i_j \in I$ and the $v_j \in V$ implies $rt = \sum_{j=1}^{n}(i_j t)v_j \in V$ (since each $i_j t \in It \subseteq V$), a contradiction. The fact that it suffices to consider only those V within the fraction field of R/P follows from the observation that one may replace V by its intersection with that field.

The proofs of (c) and (d) in the Noetherian case are precisely the same, making use of the parnthetical comment about the Noetherian case given in the paragraph above.

To prove (e) first note that the expansion of I to S_i is generated by u_i, since $u_j = \frac{u_j}{u_i}u_i$, and so $IT_i = u_iT_i$ as well. If $r \in \overline{I}$ then $r \in IT_i$ by the preceding Lemma. Now suppose instead that we assume that $r \in IT_i$ for every i instead. Consider any inclusion $R \subseteq V$, where V is a valuation domain. Then in V, the image of one of the u_j, say u_i, divides all the others, and so we can choose i such that $S_i \subseteq V$. Since V is normal, we then have $T_i \subseteq V$ as well, and then $r \in IT_i$ implies that $r \in IV$. Since this holds for all valuation domains containing R, $r \in \overline{I}$.

Finally, it will suffice to prove the "only if" part of (f) and the "if" part of (g). If $I = (0)$ we may choose $c = 1$, and so we assume that $I \neq 0$. Suppose that $J = I + Rr$ and choose $h \in \mathbb{N}$ so that $J^{h+1} = IJ^h$, so that $J^{n+h} = I^nJ^h$ for all $n \in \mathbb{N}$: see parts (a) and (c) on the second page of the Lecture Notes from September 13. Choose c to be a nonzero element of J^h. Then, for all n, $cr^n \in J^{n+h} = I^nJ^h \subseteq I^n$, as required.

Now suppose that c is a nonzero element such that $cr^n \in I^n$ for arbitrarily large values of n. If $r \notin \overline{I}$ we can choose a discrete valuation v such that the value of v on r is smaller than the value of v on any element of I: then $v(r) + 1 \leq v(u)$ for all $u \in I$. Choose $n > v(c)$. Then $nv(r) + n \leq v(w)$ for all $w \in I^n$. But if we take $w = cr^n$ we have $v(w) = v(c) + nv(r) < nv(r) + n \leq v(w)$, a contradiction. $\square$

For those familiar with the theory of schemes, we note that the condition in part (e) can be described in scheme-theoretic terms. There is a scheme, the *blow-up* Y of $X = \operatorname{Spec} R$ along the closed subscheme defined by I, which has a finite open cover by open affines corresponding to the affine schemes $\operatorname{Spec} S_i$. The normalization Y' of Y, i.e., the *normalized blow-up*, has a finite open cover by the open affines $\operatorname{Spec} T_i$. I corresponds to a sheaf of ideals on X, which pulls back (locally, via expansion) to a sheaf of ideals $\mathcal{J}$ on Y'. The integral closure of I is then the ideal of global sections of $\mathcal{J}$ intersected with R.

Discussion: the notion of analytic spread. Let (R, m, K) be a local ring and let $J \subseteq m$ be an ideal of R. It is of interest to study the least integer a such that J is integral over an ideal generated by a elements. If $I \subseteq J$ and J is integral over I, then I is called a *reduction* of J. Thus, we are interested in the minimum number of generators of a reduction.

We recall that the *associated graded ring*, denoted $\mathrm{gr}_I(R)$, of R with respect to the ideal I is the $\mathbb{N}$-graded ring

$$R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots ,$$

so that the k th graded piece is I^k/I^{k+1}. The multiplication is such that if $u \in I^j$ represents a j-form $\overline{u} \in I^j/I^{j+1}$ and $v \in I^k$ represents a k-form $\overline{v} \in I^k/I^{k+1}$, then uv represents the product $\overline{u}\,\overline{v} \in I^{j+k}/I^{j+k+1}$. This ring is generated by its forms of degree 1: moreover, given a set of generators of I as an ideal, the images of these elements in I/I^2 generate $\mathrm{gr}_I(R)$ as an (R/I)-algebra. Thus, if R is Noetherian, so is $\mathrm{gr}_I(R)$.

Now suppose that $(R,\, m,\, K)$ is a local ring, and view $K = R/m$ as an R-algebra. The ring $K \otimes_R \mathrm{gr}_I(R)$ is a finitely generated K-algebra. We may also write this as

$$K \oplus I/mI \oplus I^2/mI^2 \oplus \cdots \oplus I^n/mI^n \oplus \cdots .$$

The *analytic spread* of $I \subseteq R$ is defined to be the Krull dimension of the ring $K \otimes_R \mathrm{gr}_I(R)$. This ring is generated over K by its forms of degree 1, and the images in I/mI of any set of generators for I as an ideal generate $K \otimes \mathrm{gr}_I(R)$ as a K-algebra. We use $\mathfrak{an}(I)$ to denote the analytic spread of I. Our next main objective is to prove:

Theorem. *Let $(R,\, m,\, K)$ be a local ring and J a proper ideal. Then the number of generators of any reduction of J is at least $\mathfrak{an}(J)$, and if K is infinite, J has a reduction with $\mathfrak{an}(J)$ generators.*