Math 711: Lecture of September 18, 2006

We have already noted that when $(R,\, m,\, K)$ is a local ring and $i \subseteq m$ an ideal we may identify

$$K \otimes_R \mathrm{gr}_I(R) \cong R/m \oplus I/mI \oplus I^2/mI^2 \oplus \cdots \oplus I^n/mI^n \oplus \cdots .$$

S is called a *standard graded* A-algebra if S is $\mathbb{N}$-graded with $S_0 = A$ and the 1-forms S_1 of S generate S as an A-algebra. If S is a standard graded K-algebra, where K is a field, then R has a unique homogeneous maximal ideal $\mathfrak{m} = \bigoplus_{n=1}^{\infty} S_n$, the K-span (and even the span as an abelian group) of all elements of positive degree.

We note as well that if $R[It] \subseteq R[t]$ is the Rees ring, then

$$(R/I) \otimes_R R[It] \cong R[It]/IR[It] = \frac{R + It + I^2t^2 + \cdots + I^nt^n + \cdots}{I + I^2t + I^3t^2 + \cdots + I^{n+1}t^n + \cdots},$$

and it is quite straightforward to identify this with $\mathrm{gr}_I R$.

Since $(R/m) \otimes_R (R/I) \cong R/I$, it follows that

$$K \otimes_R \mathrm{gr}_I(R) \cong (R/m) \otimes_R \big((R/I) \otimes_R R[It]\big) \cong \big(R/m) \otimes_R (R/I)\big) \otimes_R R[It] \cong K \otimes_R R[It],$$

so that we may also view $K \otimes_R \mathrm{gr}(R)$ as $K \otimes_R R[It]$.

We give two preliminary results. Recall that in Nakayama's Lemma one has a *finitely generated module* M over a ring $(R,\, m)$ with a unique maximal ideal, i.e., a quasilocal ring. The lemma states that if $M = mM$ then $M = 0$. By applying the result to M/N, one can conclude that if M is finitely generated (or finitely generated over N), and $M = N + mM$, then $M = N$. In particular, elements of M whose images generate M/mM generate M: if N is the module they generate, we have $M = N + mM$. Less familiar is the homogeneous form of the Lemma: it does not need M to be finitely generated, although there can be only finitely many negative graded components (the detailed statement is given below).

First recall that if H is an additive semigroup with 0 and R is an H-graded ring, we also have the notion of an H-graded R-module M: M has a direct sum decomposition

$$M = \bigoplus_{h \in H} M_h$$

as an abelian group such that for all $h, k \in H$, $R_h M_k \subseteq M_{h+k}$. Thus, every M_h is an R_0-module. A submodule N of M is called graded (or homogeneous) if

$$N = \bigoplus_{h \in H} (N \cap M_h).$$

An equivalent statement is that the homogeneous components in M of every element of N are in N, and another is that N is generated by forms of M.

Note that if we have a subsemigroup $H \subseteq H'$, then any H-graded ring or module can be viewed as an H'-graded ring or module by letting the components corresponding to elements of $H' - H$ be zero.

In particular, an $\mathbb{N}$-graded ring is also $\mathbb{Z}$-graded, and it makes sense to consider a $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded ring.

Nakayama's Lemma, homogeneous form. *Let R be an $\mathbb{N}$-graded ring and let M be any $\mathbb{Z}$-graded module such that $M_{-n} = 0$ for all sufficiently large n (i.e., M has only finitely many nonzero negative components). Let I be the ideal of R generated by elements of positive degree. If $M = IM$, then $M = 0$. Hence, if N is a graded submodule such that $M = N + IM$, then $N = M$, and a homogeneous set of generators for M/IM generates M.*

Proof. If $M = IM$ and $u \in M$ is nonzero homogeneous of smallest degree d, then u is a sum of products $i_t v_t$ where each $i_t \in I$ has positive degree, and every v_t is homogeneous, necessarily of degree $\geq d$. Since every term $i_t v_t$ has degree strictly larger than d, this is a contradiction. The final two statements follow exactly as in the case of the usual form of Nakayama's Lemma. $\square$

Lemma. *Let $S \to T$ be a degree preserving K-algebra homomorphism of standard graded K-algebras. Let $\mathfrak{m} \subseteq S$ and $\mathfrak{n} \subseteq T$ be the homogeneous maximal ideals. Then T is a finitely generated S-module if and only if the image of S_1 in T_1 generates an $\mathfrak{n}$-primary ideal.*

Proof. By the homogeneous form of Nakayama's lemma, T is finitely generated over S if and only if $T/\mathfrak{m}T$ is a finite-dimensional K-vector space, and this will be true if and only if all homogeneous components $[T/\mathfrak{m}T]_s$ are 0 for $s \gg 0$, which holds if and only if $\mathfrak{n}^s \subseteq \mathfrak{m}T$ for all $s \gg 0$. $\square$

Proposition. *Let (R, m, K) be a local ring. If $I \subseteq J \subseteq m$ are ideals, then J is integral over I if and only if the image of I in $J/mJ = [K \otimes_R \operatorname{gr}(R)]_1$ generates an $\mathfrak{n}$-primary ideal in $K \otimes_R \operatorname{gr}_J(R)$, where $\mathfrak{n}$ is the homogeneous maximal ideal in T.*

Proof. First note that J is integral over I if and only if $R[Jt]$ is integral over $R[It]$, and this is equivalent to the assertion that $R[Jt]$ is module-finite over $R[It]$, since $R[Jt]$ is finitely generated as an R-algebra, and, hence, as an $R[It]$-algebra.

If this holds, we have that $K \otimes_R R[Jt]$ is a finitely generated module over $K \otimes_R R[It]$, and, since the image of I generates the maximal ideal $\mathfrak{m}$ in $S = K \otimes_R \operatorname{gr}_I(R) \cong K \otimes_R R[It]$, the preceding Lemma implies that the latter statement will be true if and only if the image of I in $J/mJ = [K \otimes_R \operatorname{gr}_J(R)]_1$ generates an $\mathfrak{n}$-primary ideal in $T = K \otimes_R \operatorname{gr}_J(R)$.

The proof will be complete if we can show that when T is module-finite over S, then $R[Jt]$ is module-finite over $R[It]$. Let $j_1 \in J^{d_1}, \ldots, j_h \in J^{d_h}$ be elements whose images in $J^{d_1}/mJ^{d_1}, \ldots, J^{d_h}/mJ^{d_h}$, respectively, generate T as an S-module. We claim that

$j_{d_1}t^{d_1}, \ldots, j_{d_h}t^{d_h}$ generate $R[Jt]$ over $R[It]$. To see this, note that the fact that these elements generate T over S implies that for every N,

$$J^N = \sum_{1 \le i \le h \text{ such that } d_i \le n} I^{N-d_i} j_{d_i} + mJ^N.$$

For each fixed N, we may apply the usual form of Nakayama's Lemma to conclude that

$$J^N = \sum_{1 \le i \le h \text{ such that } d_i \le n} I^{N-d_i} j_{d_i}.$$

and so, for all N, we have

$$J^N t^N = \sum_{1 \le i \le h \text{ such that } d_i \le n} I^{N-d_i} t^{N-d_i} j_{d_i} t^{d_i},$$

which is just what we need to conclude that $j_{d_1}t^{d_1}, \ldots, j_{d_h}t^{d_h}$ generate $R[Jt]$ over $R[It]$. $\square$

The following fact is often useful.

Proposition. *Let K be an infinite field, $V \subseteq W$ vector spaces, and let $V_1, \ldots, V_h$ be vector subspaces of W such that $V \subseteq \bigcup_{i=1}^h V_i$. Then $V \subseteq V_i$ for some i.*

Proof. If not, for each i choose $v_i \in V - V_i$. We may replace V by the span of the v_i and so assume it is finite-dimensional of dimension d. We may replace V_i by $V_i \cap V$, so that we may assume every $V_i \subseteq V$. The result is clear when $d = 1$. When $d = 2$, we may assume that $V = K^2$, and the vectors $(1, c)$, $c \in K - \{0\}$ lie on infinitely many distinct lines. For $d > 2$ we use induction. Since each subspace of $V \cong K^d$ of dimension $d - 1$ is covered by the V_i, each is contained in some V_i, and, hence, equal to some V_i. Therefore it suffices to see that there are infinitely many subspaces of dimension $d - 1$. Write $V = K^2 \oplus W$ where $W \cong K^{d-2}$. The line L in K^2 yields a subspace $L \oplus W$ of dimension $d - 1$, and if $L \neq L'$ then $L \oplus W$ and $L' \oplus W$ are distinct subspaces. $\square$

Also note:

Proposition. *Let M be an $\mathbb{N}$-graded or $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded or $\mathbb{Z}$-graded Noetherian ring S. Then every associated prime of M is homogeneous. Hence, every minimal prime of the support of M is homogeneous and, in particular the associated (hence, the minimal) primes of S are homogeneous.*

Proof. Any associated prime P of M is the annihilator of some element u of M, and then every nonzero multiple of $u \neq 0$ can be thought of as a nonzero element of $S/P \cong Su \subseteq M$, and so has annihilator P as well. Replace u by a nonzero multiple with as few nonzero homogeneous components as possible. If u_i is a nonzero homogeneous component of u of degree i, its annihilator J_i is easily seen to be a homogeneous ideal of S. If $J_h \neq J_i$ we can choose a form F in one and not the other, and then Fu is nonzero with fewer homgeneous components then u. Thus, the homogeneous ideals J_i are all equal to, say, J, and clearly $J \subseteq P$. Suppose that $s \in P - J$ and subtract off all components of s that are in J, so that no nonzero component is in J. Let $s_a \notin J$ be the lowest degree component of s and u_b be the lowest degree component in u. Then $s_a u_b$ is the only term of degree $a+b$ occurring in $su = 0$, and so must be 0. But then $s_a \in \mathrm{Ann}_S\, u_b = J_b = J$, a contradiction. $\square$

Corollary. *Let S be a standard graded K-algebra of dimension d with homogeneous maximal ideal $\mathfrak{m}$. Then there are forms $L_1, \ldots, L_d$ of degree 1 in R_1 such that $\mathfrak{m}$ is the radical of $(L_1, \ldots, L_d)S$.*

Proof. The minimal primes of a graded algebra are homogenous, and $\dim(S)$ is the same as $\dim(S/P)$ for some minimal prime P of R. Then $P \subseteq \mathfrak{m}$, and

$$\dim(S) = \dim(S/P) = \dim(S/P)_{\mathfrak{m}} \leq \dim S_{\mathfrak{m}} \leq \dim(S),$$

so that $\dim(S) = \dim(S_{\mathfrak{m}}) = \mathrm{height}\,\mathfrak{m}$. If $\dim(S) = 0$, $\mathfrak{m}$ must be the unique minimal prime of S, and therefore is itself nilpotent. Otherwise, S_1 cannot be contained in the union of the minimal primes of S, or the Proposition just above would imply that it is contained in one of them, and S_1 generates $\mathfrak{m}$. Choose $L_1 \in S_1$ not in any minimal prime, and then $\dim(S/L_1) = d-1$. Use induction. If $L_1, \ldots, L_k$ have been chosen in S_1 such that $\dim\big(S/(L_1, \ldots, L_k)S\big) = d-k < d$, choose $L_{k+1} \in S_1$ not in any minimal prime of $(L_1, \ldots, L_k)S$ (if S_1 were contained in one of these, $\mathfrak{m}$ would be, and it would follow that $\mathrm{height}\,\mathfrak{m} \leq k$, a contradiction). Thus, we eventually have $L_1, \ldots, L_d$ such $\dim(S/\big(L_1, \ldots, L_d)S\big) = 0$, and then by the case where $d = 0$ we have that $\mathfrak{m}$ is nilpotent modulo $(L_1, \ldots, L_d)S$. $\square$

We are now ready to prove the result that we have been aiming for:

Theorem. *Let (R, m, K) be local and $J \subseteq R$ an ideal. Then any reduction I of J has at least $\mathfrak{an}(J)$ generators. Moreover, if K is infinite, there is a reduction with $\mathfrak{an}(J)$ generators.*

Proof. The problem of giving $i_1, \ldots, i_a \in J$ such that J is integral over $(i_1, \ldots, i_a)R$ is equivalent to giving a elements of J/mJ that generate an $\mathfrak{m}$-primary ideal of $S = K \otimes_R \mathrm{gr}_J(R)$, where $\mathfrak{m}$ is the homogeneous maximal ideal of S. Clearly, we must have $a \geq \dim(S) = \mathfrak{an}(J)$. If K is infinite, the existence of suitable elements follows from the Corollary just above. $\square$

Discussion. If (R, m, K) is local and t is an indeterminate over R, let $R(t)$ denote the localization of the polynomial ring $R[t]$ at $mR[t]$. Then $R \to R(t)$ is a faithfully flat map of local rings of the same dimension, and the maximal ideal of $R(t)$ is $mR(t)$ while the residue class field of $R(t)$ is $K(t)$. If $J \subseteq m$, $\mathfrak{an}(J)$ is the least number of generators of an ideal over which $JR(t)$ is integral. In fact, $K(t) \otimes \mathrm{gr}_{JR(t)} R(t) \cong K(t) \otimes_K \big(K \otimes_R \mathrm{gr}_J(R)\big)$, so that $\mathfrak{an}(J) = \mathfrak{an}\big(JR(t)\big)$.

Remark. If I and J are any two ideals of any ring R, $\overline{I}\,\overline{J} \subseteq \overline{IJ}$. There are many ways to see this. E.g., if $r \in \overline{I}$, $s \in \overline{J}$ and $R \to V$ is any homomorphism to a valuation domain, then $r \in IV$ and $s \in JV$, whence $rs \in (IV)(JV) = (IJ)V$. Thus, $rs \in \overline{IJ}$ for every such r and s, and the elements rs generate $\overline{I}\,\overline{J}$. $\square$

We shall prove below that if R is local and J is any proper ideal, then $\dim(R) = \dim(\mathrm{gr}_J R)$. Assume this for the moment. It then follows that $\dim\bigl(K \otimes_R \mathrm{gr}_J(R)\bigr) \leq \dim(R)$. We define the *big height* of a proper ideal J of a Noetherian ring to be the largest height of any minimal prime of J. (The height is the smallest height of any minimal prime of J.) We then have:

Proposition. *For any proper ideal J of a local ring (R, m, K), the analytic spread of J lies between the big height of J and $\dim(R)$.*

Proof. That $\mathfrak{an}(I) \leq \dim(R)$ follows from the discussion above, once we have shown that $\dim(R) = \dim(\mathrm{gr}_J R)$. Let P be a minimal prime of J. We want to show that $\mathrm{height}\, P \leq \mathfrak{an}(J)$. After replacing R by $R(t)$, if necessary, we have that J is integral over an ideal I with $a = \mathfrak{an}(J)$ generators. Then J is contained in the radical of I. In R_P we have that P is the radical of JR_P, since P is a minimal prime of J, and so is contained in the radical of IR_P. Thus, $\mathrm{height}\, P \leq a = \mathfrak{an}(J)$, as required. $\square$

Corollary. *If K is infinite, every proper ideal J of a local ring (R, m, K) of Krull dimension d is integral over an ideal generated by at most d elements.* $\square$

Proposition. *Let (R, m, K) be local, and J a proper ideal. Then for every positive integer n, the ideals J and J^n have the same analytic spread.*

Proof. The Rees ring of J^n may be identified with $R[J^n t^n]$ since t^n is an indeteminate over R, and this is a subring of $R[Jt]$ over which the larger ring is module-finite, since the n th power of any element of Jt is in $R[J^n t^n]$. The injectivity is retained when we apply $K \otimes_R \_$, since tensor commutes with direct sum, and the module-finite property continues to hold as well. It follows that $K \otimes \mathrm{gr}_J(R)$ is a module-finite extension of $K \otimes_R \mathrm{gr}_{J^n} R$, and so these two rings have the same dimension. $\square$

In general, if X is a matrix and B is a ring, $B[X]$ denotes the ring generated over B by the entries of X. We frequently use this notation when these entries are indeterminates, in which case $B[[X]]$ denotes the formal power series ring over B in which the variables are the entries of X. If $M = \bigl(r_{ij}\bigr)$ is a matrix over a ring R and t is a nonnegative integer, $I_t(M)$ denotes the ideal of R generated by the size t minors of M. By convention, this ideal is R if $t = 0$ and is (0) if t is strictly larger than either of the dimensions of the matrix M.

Example. Let $I \subseteq (R, m, K)$ and let r be a nonzerodivisor. Then $R[It] \cong R[rIt]$: in fact rt is algebraically independent of R, so that there is an R-isomorphism $R[t] \to R[rt]$ mapping $t \mapsto rt$, and this induces an R-isomorphism $RI[t] \cong R[rIt]$. It follows that

$K \otimes_R R[It] \cong K \otimes_R R[rIt]$, and so $\mathfrak{an}(I) = \mathfrak{an}(Ir)$. $Ir \subseteq rR$ which has analytic spread one. If $I = m$, or if I is m-primary, the analytic spread of I and of Ir is $\dim(R)$. Thus, the smaller of two ideals may have a much larger analytic spread than the larger ideal.

Example. Let K be a field and let $X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}$ be a $2 \times n$ matrix of formal indeterminates over K. Let $K[X]$ be the polynomial ring in the entries of X, and let $A = K[X]/I_2(X)$. This ring is known to be a normal ring with an isolated singularity of dimension $n+1$. One can see what the dimension is as follows: we may tensor with algebraic closure of K without changing the dimension (this produces an integral extension), and so we may assume that K is algebraically closed. The algebraic set Z in $\mathbb{A}^{2n}$ defined by $I_2(X)$ corresponds to $2 \times n$ matrices of rank at most one. We can map $\mathbb{A}^n \times \mathbb{A}^1$ to Z by sending (v, c) to the matrix whose first row is v and whose second row is cv. This map is not onto, but its image contains the open set consisting of matrices whose first row is not 0. It follows that the dimension of Z is $n + 1$. (This ring is also known to be Cohen-Macaulay. Cf. [M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. of Math. **93** (1972), 1020–1058].) The same properties hold if we localize A at the ideal generated by the entries of X (and if we complete). Call the local ring obtained S. Let P be the prime ideal $(x_1, \dots, x_n)R$. Then $A = S/P$ is either the localization of $K[y_1, \dots, y_n]$ at $(y_1, \dots, y_n)$ or its completion. In any case, R/P has dimension n, so that P is a height one prime of S. But its analytic spread is n. In fact, $\mathrm{gr}_P(S)$ has the form $A[u_1, \dots, u_n]$ where the u_i satisfy the relations $y_i u_j - y_j u_i = 0$. If we kill only these relations we get a domain of dimension $n + 1$ that maps onto $\mathrm{gr}_P(S)$. Since $\mathrm{gr}_P(S)$ has the same dimension as S (we have not proved this yet, but will shortly), the map onto $\mathrm{gr}_P(S)$ cannot have a nonzero kernel, i.e., it is an isomorphism. But then $K \otimes \mathrm{gr}_P(S) \cong K[u_1, \dots, u_n]$ has dimension n.

We next want to prove the assertion that the local ring (R, m, K) and the ring $\mathrm{gr}_J R$ have the same dimension. In order to do so, we review the dimension formula. Recall that a Noetherian ring R is *catenary* if for any two prime ideals $P \subseteq Q$, any two saturated chains of prime ideals joining P to Q have the same length. Localizations and homomorphic images of catenary rings are clearly catenary. R is *universally catenary* if every polynomial ring in finitely many variables over R is catenary. It is equivalent to assert that every algebra essentially of finite type over R is catenary. A ring is called *Cohen-Macaulay* if in each of its local rings some (equivalently, every) system of parameters is a regular sequence. Cohen-Macaulay rings are catenary and, therefore, universally catenary, since a polynomial ring over a Cohen-Macaulay ring is Cohen-Macaulay. Regular rings are Cohen-Macaulay, and, hence, universally catenary. A complete local ring is a homomorphic image of a regular ring and so is also universally catenary. If $\mathcal{F} \subseteq \mathcal{G}$ are fields, $\mathrm{tr.\,deg.}(\mathcal{G}/\mathcal{F})$ denotes the transcendence degree over $\mathcal{G}$ over $\mathcal{F}$.

Theorem (dimension formula). *Let $R \subseteq S$ be Noetherian domains such that S is finitely generated over R, and call the fraction fields $\mathcal{F}$ and $\mathcal{G}$, respectively. Let Q be a prime ideal of S lying over P in R. Let K and L be the residue class fields of R_P and S_Q,* *respectively. Then*

$$\operatorname{height} Q - \operatorname{height} P \leq \operatorname{tr.deg.}(\mathcal{G}/\mathcal{F}) - \operatorname{tr.deg.}(L/K),$$

with equality if R is universally catenary.