Math 711: Lecture of September 20, 2006

We shall soon return to our treatment of the dimension formula, which was stated in the Lecture of September 18, but we first want to make some additional remarks about the behavior of analytic spread.

Theorem. *Let K be a field, and T a finitely generated $\mathbb{N}$-graded K-algebra with $T_0 = K$. Let $\mathcal{M}$ be the homogenous maximal ideal of T. Let $F_1, \ldots, F_s$ be homogeneous polynomials of the same positive degree d in T, and let $I = (F_1, \ldots, F_s)T$. Then $\mathfrak{an}(IT_{\mathcal{M}})$ is the Krull dimension of the ring $K[F_1, \ldots, F_s] \subseteq T$, and hence is the same as the maximum number of algebraically independent elements in $K[F_1, \ldots, F_s]$. over K.*

Proof. We shall show that $K[F_1, \ldots, F_s] \cong K \otimes_{T_{\mathcal{M}}} \mathrm{gr}_I(T_{\mathcal{M}})$. We view the latter is

$$K \otimes_{T_{\mathcal{M}}} T_{\mathcal{M}}[IT_{\mathcal{M}}t] \cong (K \otimes_T T[It])_{\mathcal{M}}$$

and the ring on the right is the same as $K \otimes_T T[It]$, because elements of $T - \mathcal{M}$ map to units in K and so already are invertible in this ring. Note that K here is $T/\mathcal{M}$, and so this ring is also the same as $T[It]/\mathcal{M}T[It]$.

Now there is a map $K[F_1, \ldots, F_s] \to T[It]$ that sends $F_j \mapsto F_j t$, $1 \le j \le s$. To see that this is well-defined, note that the ideal of relations on the F_j over K is homogeneous. Thus, it suffices to see that if $H \in K[Y_1, \ldots, Y_s]$ is a *homogeneous* polynomial of degree μ such that $H(F_1, \ldots, F_s) = 0$, then $H(F_1t, \ldots, F_st) = 0$. But the left hand side is $t^\mu H(F_1, \ldots, F_d) = t^\mu \cdot 0 = 0$. We then get a composite map

$$K[F_1, \ldots, F_s] \to T[It] \twoheadrightarrow T[It]/\mathcal{M}T[It].$$

This map is clearly surjective, since the image of T in the quotient is K and It is generated by the F_jt. We need only prove that the kernel is 0. It is homogeneous: let G be an element of the kernel that is homogeneous of degree h in $F_1, \ldots, F_s$. Then G has degree hd in $x_1, \ldots, x_n$. If G is in the kernel then Gt^h is in $\mathcal{M}I^ht^h$, and $G \in \mathcal{M}I^h$. However, all nonzero elements of this ideal have components of degree at least $hd + 1$ in $x_1, \ldots, x_n$, a contradiction unless $G = 0$.

The final statement is a general characterization of Krull dimension in finitely generated K-algebras. $\square$

Remark. This result gives another way to compute the analytic spread of the height one prime in a determinantal ring analyzed in the last Example (beginning at the bottom of p. 5) in the Lecture Notes of September 18. It is immediate that the analytic spread is n.

Example. The Example discussed in the Remark just above shows that height one primes that have arbitrarily large analytic spread. In a regular local ring a height one prime is principal, and so its analytic spread is 1. But there are height two primes of arbitarily large analytic spread. Let X be an $n \times (n+1)$ matrix of indeterminates over a field K and let P be the ideal generated by the size n minors of X in the polynomial ring $K[X]$. Then the analytic spread of P in $K[X]_{\mathcal{M}}$, where $\mathcal{M}$ is generated by the entries of X, is $n+1$ by the Theorem above, for the minors of algebraically independent over K. (This is true even if we specialize the leftmost $n \times n$ submatrix to be yI_n. The minors are y^n and, up to sign, the products $y^{n-1}x_{i,n+1}$, $1 \leq i \leq n$.) These primes have height two: the algebraic set of $n \times (n+1)$ matrices of rank at most $n-1$ has dimension n^2+n-2. (On the open set where the first $n-1$ rows are algebraically independent, the space consisting of choices for the first $n-1$ rows has dimension $(n-1)(n+1)$; the choices for the final row are linear combinations of the first $n-1$ rows, and are parametrized by $\mathbb{A}^{n-1}$, giving dimension $n^2-1+(n-1)$.)

Recall that a map of quasilocal rings $h : (R, m) \to (S, n)$ is called *local* if $h(m) \subseteq n$. (The map of a local ring onto its residue class field is local, while the inclusion of a local domain that is not a field in its fraction field is *not* local.)

Proposition. *Let* (R, m, K) *be local.*

(a) *If* $h : (R, m, K) \to (S, n, L)$ *is a local homomorphism, and* $I \subseteq m$ *is an ideal of* R*, then* $\mathfrak{an}(I) \geq \mathfrak{an}(IS)$*.*

(b) *If* I *and* J *are proper ideals of* R*, then* $\mathfrak{an}(I+J) \leq \mathfrak{an}(I) + \mathfrak{an}(J)$*.*

(c) *Let* I *and* J *are proper ideals of* R*. If either* $\mathfrak{an}(I)$ *or* $\mathfrak{an}(J)$ *is 0, then* $\mathfrak{an}(IJ) = 0$*. If the analytic spreads are positive,* $\mathfrak{an}(I\,J) \leq \mathfrak{an}(I) + \mathfrak{an}(J) - 1$*.*

Proof. We replace $R \to S$ by $R(t) \to S(t)$ if necessary, and the ideals considered by their expansions. We may therefore assume the residue class fields are infinite.

For part (a), if I is integral over an ideal I_0 with $a = \mathfrak{an}(I)$ generators, then IS is integral over I_0S.

For part (b) simply note that if I_0 is as above and J is integral over J_0 with $b = \mathfrak{an}(J)$ generators, then $I+J$ is integral over $I_0 + J_0$, which has at most $a+b$ generators.

To prove part (c), first note that the analytic spread of I is 0 if and only if I consists of nilpotents. Thus, if either a or b is 0, then IJ consists of nilpotents and $\mathfrak{an}(I\,J) = 0$ as well. Now suppose that both analytic spreads are positive and that I_0 and J_0 are as above. Map the polynomial ring $T = \mathbb{Z}[X_1, \ldots, X_a, Y_1, \ldots, Y_b] \to R$ so that $(X_1, \ldots, X_a)T$ maps onto I_0 and $(Y_1, \ldots, Y_b)T$ maps onto J_0. Since IJ is integral over I_0J_0, it suffices to show that $\mathfrak{an}(I_0J_0) \leq a+b-1$. Let $\mathcal{M}$ be the inverse image of m in T. Then $\mathcal{M}$ is a prime ideal of T that is either $(X_1, \ldots, X_a, Y_1, \ldots, Y_b)T$ or $pT + (X_1, \ldots, X_a, Y_1, \ldots, Y_b)T$. Let $A = T_{\mathcal{M}}$. Let $\mathcal{I} = (X_1, \ldots, X_a)A$ and $\mathcal{J} = (Y_1, \ldots, Y_b)A$. Then we have an induced local map $T_{\mathcal{M}} \to R$ such that $\mathcal{I}R = I_0$ and $\mathcal{J}R = J_0$. By part (a), it will suffice to show that $\mathfrak{an}(\mathcal{I}\,\mathcal{J}) \leq a+b-1$.

Let $\mathfrak{A}$ denote the ideal $(X_1, \ldots, X_a, Y_1, \ldots, Y_b) \subseteq T$, and let $\mathfrak{B}$ denote the ideal $(X_1, \ldots, X_a)(Y_1, \ldots, Y_b)T$. There are two cases. First suppose that $\mathcal{M} = \mathfrak{A}$. Then

$T_{\mathcal{M}}$ contains the rational numbers, and may be viewed instead as the localization of the polynomial ring $\mathbb{Q}[X_1, \ldots, X_a, Y_1, \ldots, Y_b]$ at $(X_1, \ldots, X_a, Y_1, \ldots, Y_b)$. Then, since the elements X_iY_j are forms of the same degree, the Theorem above applies, and the analytic spread is the transcendence degree of $\mathbb{Q}[X_iY_j : 1 \le i \le a, 1 \le j \le b]$ over $\mathbb{Q}$. But the fraction field of this domain is generated by the elements $X_1Y_1, \ldots, X_1Y_b$ and the elements X_j/X_1, $2 \le i \le a$, since $(X_i/X_1)(X_1Y_j) = X_iY_j$ (which also shows that each X_j/X_1 is in the fraction field). These $b+(a-1)$ elements are easily seen to be algebraically independent. Exactly the same calculation of transcendence degree if $\mathbb{Q}$ is replaced by any other field κ.

In the remaining case, $\mathcal{M} = \mathfrak{A} + pT$. In this case, note that since $\mathfrak{B}^n$ and $\mathfrak{B}^{n+1}$ are both free $\mathbb{Z}$-modules spanned by the monomials in $X_1, \ldots, X_a, Y_1, \ldots, Y_b$ that they contain, each $\mathfrak{B}^n/\mathfrak{B}^{n+1}$ is free over $\mathbb{Z}$. It follows that p is not a zerodivisor on $\mathrm{gr}_{\mathfrak{B}}T$. Let $\kappa = T/\mathcal{M} \cong \mathbb{Z}/p\mathbb{Z}$. Then

$$\kappa \otimes_T \mathrm{gr}_{\mathfrak{B}}T \cong \kappa \otimes_{T/pT} \big((\mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{gr}_{\mathfrak{B}}T\big).$$

Let

$$\overline{T} = T/pT \cong \kappa[X_1, \ldots, X_a,\ Y_1, \ldots, Y_b],$$

and $\overline{\mathfrak{B}} = \mathfrak{B}\overline{T}$, Because p is not a zerodivisor on $\mathrm{gr}_{\mathfrak{B}}(T)$, we have that

$$(\mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{gr}_{\mathfrak{B}}T \cong \mathrm{gr}_{\overline{\mathfrak{B}}}\overline{T},$$

and this is the ring whose dimension we need to calculate: as in the proof of the Theorem above, localization at $\mathcal{M}$ has no effect on this ring, since the image of $T - \mathcal{M}$ consists of units in κ. We are now in the same situation as in the first case, except that we are working with $\kappa[X_1, \ldots, X_a, Y_1, \ldots, Y_b]$ instead of $\mathbb{Q}[X_1, \ldots, X_a, Y_1, \ldots, Y_b]$. $\square$

We are now ready to continue with our treatment of the dimension formula, stated in the Lecture of September 18. Recall that we are assuming that $R \subseteq S$ are Noetherian domains with fraction fields $\mathcal{F}$ and $\mathcal{G}$ respectively, that Q is a prime ideal of S lying over P in R, that $K = R_P/PR_P$, and that $L = S_Q/QS_Q$. We must show that

$$\mathrm{height}\, Q - \mathrm{height}\, P \le \mathrm{tr.\,deg.}\mathcal{G}/\mathcal{F} - \mathrm{tr.\,deg.}L/K,$$

with equality of R is universally catenary. Equality also holds if S is a polynomial ring over R.

Before beginning the proof, we make the following observation. Let P_0 be a prime ideal of a local domain D. In general, $\dim(D/P_0) \le \dim(D) - \mathrm{height}\, P_0$, while equality holds if D is catenary. The inequality, which is equivalent to the statement that $\dim(D) \ge \mathrm{height}\, P_0 + \dim(D/P_0)$, follows from the following observation. We can "splice" a saturated chain of primes of length $k = \dim(D/P_0)$ ascending from P_0 to the maximal ideal m of D (corresponding to a chain of primes of length k in D/P_0) with a chain of primes of length h descending from P_0 to (0). This yields a chain of saturated primes from m to (0) in D

that has length $h+k$. If, moreover, D is catenary then all saturated chains from m to (0) have the same length, and this is $\dim(D)$, so that $h+k=\dim(D)$.

Proof of the dimension formula. By adjoining generators of S to R one at a time, we can construct a chain of rings

$$R = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n$$

such that for each i, $0 \le i \le n$, we have that S_{i+1} is generated over S_i by one element. Let $Q_i = Q \cap S_i$ for each i. Note that when R is universally catenary, every S_i is universally catenary. It will suffice to prove the dimension formula (whether the inequality or the equality) for each inclusion $S_i \subseteq S_{i+1}$. When we add the results, each term associated with S_i for i different from 0 and n occurs twice with opposite signs. The intermediate terms all cancel, and we get the required result.

We henceforth assume that $S = R[x]$, where x need not be an indeterminate over R. By replacing R and S by R_P and $R_P \otimes_R S$, we may assume that (R, P, K) is local. We consider two cases, according as whether x is transcendental or algebraic over R.

Case 1. x is transcendental over R. Then the primes of $S = R[x]$ lying over P correspond to the primes of $R[x]/PR[x] \cong K[x]$, a polynomial ring in one variable. There are two subcases.

Subcase 1a. Q corresponds to the prime ideal (0) in $K[x]$, i.e., $Q = PR[x]$. In this case $S_Q \cong R(x)$ has the same dimension as R, so that $\operatorname{height} Q = \operatorname{height} P$. We have that $\operatorname{tr.deg.}(\mathcal{G}/\mathcal{F}) = 1$, and $L \cong K(x)$, so that $\operatorname{tr.deg.}(L/K) = 1$ as well. Since $0 = 1 - 1$, we have the required equality whether R is universally catenary or not.

Subcase 1b. Q is generated by $PR[x]$ and a monic polynomial g of positive degree whose image $\overline{g}$ mod P is irreducible in $K[x]$. The height of Q is evidently has height $\operatorname{height} P + 1$: a system of parameters for P together with g will give a system of parameters for $R[x]_Q$. The left hand side of the inequality is therefore 1, while the right hand side is $1-0$, because $L \cong K[x]/(\overline{g})$. Again, we have the required equality whether R is universally catenary or not.

Case 2. x is algebraic over R. Let X be an indeterminate and map $R[X] \twoheadrightarrow R[x] = S$ as R-algebras by sending $X \mapsto x$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}[X] & \longrightarrow & \mathcal{G} \\ \uparrow & & \uparrow \\ R[X] & \longrightarrow & S \end{array}$$

where the horizontal arrows are surjective and the vertical arrows are inclusions. By hypothesis, the top horizontal arrow has a kernel, which will be the principal ideal generated by a monic polynomial h of positive degree: the minimal polynomial of x over $\mathcal{F}$. The kernel P_0 of $R[X] \twoheadrightarrow S$ may therefore be described as $h\mathcal{F}[X] \cap R[X]$. We claim that P_0 is a height one prime of $R[X]$. To see this, we calculate $R[X]_{P_0}$. Since $R \subseteq S$, P_0 does not meet R, and $R - \{0\}$ becomes invertible in R_{P_0}. Thus, R_{P_0} is the localization of $\mathcal{F}[X]$ at the expansion of P_0, which is $h\mathcal{F}[x]$, and is a one-dimensionsial ring. Let $\mathcal{Q}$ denote the inverse image of Q in $R[X]$. Then $\mathcal{Q}$ contains P and, in fact, lies over P. It also contains P_0. There are again two subcases, depending on what $\mathcal{Q}$ is.

Subcase 2a. $\mathcal{Q} = PR[X]$. In this subcase the right hand side of the dimension formula is $0-1$. The height of $\mathcal{Q}$ is the same as height P, and killing P_0 decreases it at least by 1 as required. If R is universally catenary it decreases by exactly 1.

Subcase 2b. $\mathcal{Q}$ has the form $PR[X] + fR[X]$, where $f \in R[X]$ is monic of positive degree and irreducible mod P. The right hand side of the dimension formula is $0-0$. The height of $\mathcal{Q}$ is height $P+1$. Killing P_0 decreases it by least 1, and by exactly 1 in the universally catenary case. $\square$

Remark. If S is a polynomial ring over R, we can choose the chain so that S_{i+1} is always a polynomial ring in one variable over S_i. We are always in Case 1 of the proof, and so equality holds in the dimension formula without assuming that R is universally catenary.