Math 711: Lecture of September 22, 2006

Let R be any ring and $I \subseteq R$ any ideal. By the *extended Rees ring* or *second Rees ring* of I over R we mean the ring $R[It, 1/t] \subseteq R[t]$. In this context we shall standardly write v for $1/t$. Note that if I is proper, v is *not* a unit of $R[It,\, v]$. This ring is $\mathbb{Z}$-graded. Written out as a sum of graded pieces

$$R[It,\, v] = \cdots + Rv^k + \cdots + Rv^2 + Rv + R + It + I^2t^2 + \cdots + I^nt^n + \cdots .$$

The element v generates a homogeneous principal ideal, and

$$vR[It,\, v] = \cdots + Rv^k + \cdots + Rv^2 + Rv + I + I^2t + I^3t^2 + \cdots + I^{n+1}t^n + \cdots .$$

From this it follows easily that $R[It,\, v]/(v) \cong \mathrm{gr}_I R$. There is a composite surjection

$$R[It,\, v] \twoheadrightarrow \mathrm{gr}_I R \twoheadrightarrow R/I.$$

When I is the unit ideal of R we have that $R[It, v] = R[t,\, t^{-1}]$.

When $(R,\, m,\, K)$ is local and I is proper we further have a composite surjection

$$R[It,\, v] \twoheadrightarrow R/I \to R/m = K,$$

and the kernel is a maximal ideal $\mathcal{M}$ of $R[It,\, v]$. Explicitly,

$$\mathcal{M} = \cdots + Rv^k + \cdots + Rv^2 + Rv + m + It + I^2t^2 + \cdots + I^nt^n + \cdots .$$

Theorem. *Let $(R,\, m,\, K)$ be local, let $I \subseteq R$ be proper, and let $R[It,\, v]$ and $\mathcal{M}$ be as in the paragraphs just above,*

(a) *The Krull dimension of $R[It,\, 1/t]$ is $\dim(R) + 1$, and this is the height of $\mathcal{M}$.*

(b) $\dim\big(\mathrm{gr}_I(R)\big) = \dim(R)$.

Proof. Let

$$\mathcal{P} = \cdots + mv^k + \cdots + mv^2 + mv + m + It + I^2t^2 + \cdots + I^nt^n + \cdots ,$$

which is the contraction of $mR[t, 1/t]$ to $R[It,\, v]$. Then $\mathcal{P} \subseteq \mathcal{M}$ and $R[It,\, v]/\mathcal{P} \cong K[v]$, a polynomial ring in one variable over a field. The height of $\mathcal{P}$ is the same as the height of m: when we localize at $\mathcal{P}$ in $R[It,\, v]$, v becomes invertible, so that $t = 1/v$ becomes an element of the localized ring. But $R[It,\, v][t] = R[t,\, v]$, and the expansion of $\mathcal{P}$ is $mR[t, 1/t]$. The localization at the expansion is just $R(t)$ (note that when we localize $R[t]$ at $mR[t]$, v becomes an element of the ring), which we already know has the same dimension as R. Thus, $\mathrm{height}\,\mathcal{P} = \dim(R)$. Since $\mathcal{M} = \mathcal{P} + vR[It,\, v]$ is strictly larger than $\mathcal{P}$, we have that height $\mathcal{M} \geq \dim(R)+1$. To complete the proof of (a), it will suffice to show that $\dim(R[It,\, v]) \leq \dim(R)+1$, for then height $\mathcal{M} \leq \dim(R)+1$ as well.

We first reduce to the case where R is a domain. To do so, we want to understand the minimal primes of $S = R[It,\, v]$. If $\mathfrak{q}$ is any prime of S, it lies over some prime of R, and this prime contains a minimal prime $\mathfrak{p}$ of R. We shall show that there is a unique minimal prime $\widetilde{\mathfrak{p}}$ of S containing $\mathfrak{p}$, and it will follow that every minimal prime has the form $\widetilde{\mathfrak{p}}$. To see this, note that $\mathfrak{q}$ cannot contain v, for v is not a zerodivisor in S. Hence, $\mathfrak{q}$ corresponds via expansion to a minimal prime of S_v containing $\mathfrak{p}$. But $S_v \cong R[t, v]$, and $\mathfrak{p}R[t,\, 1/t]$ is already a minimal prime of $R[t,\, 1/t]$. It follows that $\mathfrak{q} = \mathfrak{p}R[t,\, 1/t] \cap S$, and this is the minimal prime $\widetilde{\mathfrak{p}}$. Note that $R[It,\, 1/t]/\widetilde{\mathfrak{p}}$ embeds in $(R/\mathfrak{p})[t,\, 1/t]$, and that the image is the extended Rees ring of $I(R/\mathfrak{p})$. Therefore, it suffices to show that the dimension of each of these Rees rings over a domain D obtained by killing a minimal prime of R has dimension at most $\dim(D)+1 \leq \dim(R)+1$, and we may therefore assume without loss of generality that R is a local domain.

But S is then a domain finitely generated over R. If the fraction field of R is $\mathcal{F}$, then the fraction field of S is $\mathcal{F}(t)$. If Q is any prime ideal of S, Q lies over, say, P in R, and the residue class fields of R_P and S_Q are κ_P and κ_Q respectively, then the dimension formula yields

$$\text{height}\, Q \leq \text{height}\, P + \text{tr. deg.}(\mathcal{F}(t)/\mathcal{F}) - \text{tr. deg.}(\kappa_Q/\kappa_P) \leq \text{height}\, P + 1 \leq \dim(R)+1,$$

as required.

For part (b), note that the height of $\mathcal{M}$. which is $\dim(R)+1$ drops exactly 1 when we kill the nonzerodiivisor v. This shows that $\dim\bigl(\mathrm{gr}_I(R)\bigr) \geq \dim(R)$. But killing a nonzerodivisor in a Noetherian domain of finite Krull dimension drops the dimension by at least one, so that $\dim\bigl(\mathrm{gr}_I(R)\bigr) \leq \dim(S) - 1 = \dim(R)$. $\square$

Corollary. *Let $x_1, \ldots, x_n$ be a system of parameters in a local ring (R, m, K). Let F be a homogenous polynomial of degree d in $R[X_1, \ldots, X_n]$ such that $F(x_1, \ldots, x_n) = 0$. That is, F gives a relation over R on the monomials of degree d in $x_1, \ldots, x_n$. Then all coefficients of F are in m.*

Proof. Consider the associated graded ring $\mathrm{gr}_I(R)$, where $I = (x_1, \ldots, x_n)R$. This ring is generated by the images $\overline{x}_1, \ldots, \overline{x}_n$ of $x_1, \ldots, x_n$ in $I/I^2 = [\mathrm{gr}_I(R)]_1$. Let $A = R/I$, an Artin local ring. By the preceding Theorem, $\dim\bigl(\mathrm{gr}_I(R)\bigr) = n$. But $\mathrm{gr}_I(R) = A[\overline{x}_1, \ldots, \overline{x}_n]$. Killing the maximal ideal m/I of A does not affect the dimension of this ring. It follows that quotient has dimension n, so that $K[\overline{x}_1, \ldots, \overline{x}_n]$ is a polynomial ring in $\overline{x}_1, \ldots, \overline{x}_n$. If $F(x_1, \ldots, x_n) = 0$ and has a coefficient outside m, we find the $\overline{F}(\overline{z}_1, \ldots, \overline{z}_n) = 0$ in $K[\overline{x}_1, \ldots, \overline{x}_n]$, where $\overline{F}$ is the image of F mod m and so is a nonzero polynomial in the $K[\overline{x}_1, \ldots, \overline{x}_n]$. This forces the dimension of $K \otimes_R \mathrm{gr}_I(R)$ to be smaller than n, a contradiction. $\square$

We next want to prove two consequences of the Briançon-Skoda Theorem that were stated without proof in as Corollaries at the bottom of p. 1 and the top of p. 2 of the Lecture Notes of September 6. The next result generalizes the first Corollary.

Theorem (corollary of the Briançon-Skoda Theorem). *Let R be a regular Noetherian ring of Krull dimension n and let $f_1, \dots, f_{n+1}$ be elements of R. Then*

$$f_1^n \cdots f_{n+1}^n \in (f_1^{n+1}, \dots, f_{n+1}^{n+1})R.$$

Proof. Call the product on the left g and the ideal on the right I. If $g \notin I$, then $(I + Rg)/I$ is not zero, and we can localize at a prime in its support. Therefore, we may without loss of generality that assume that (R, m, K) is a regular local ring of dimension at most n. Second, if $g \notin I$ this remains true when we replace R by $R(t)$, since $R(t)$ is faithfully flat over R. We also have that $R(t)$ and R have the same dimension. Thus, we may assume that R has an infinite residue class field. Let $h = f_1 \cdots f_n$, so that $g = h^n$. Since $h^{n+1} \in I^{n+1}$, $h \in \overline{I}$. Since $\mathfrak{an}(I) \leq \dim(R) \leq n$ and the residue class field is infinite, I is integral over an ideal I_0 with at most n generators. Then $h \in \overline{I_0}$, and it follows from the Briançon-Skoda theorem that $h^n \in I_0 \subseteq I$, as required. $\square$

We next observe:

Theorem. *Let R denote $\mathbb{C}\{\{z_1, \dots, z_n\}\}$ or $\mathbb{C}[[z_1, \dots, z_n]]$, the convergent or formal powers series ring in n variables. Let f be in the maximal ideal of R, and let I be the ideal generated by the partial derviatives $\partial f/\partial z_i$ of f. Then f is integrally dependent on I.*

Proof. We assume the result from the first Problem Set, Problem #6, that the integral closure of I is an intersection of integrally closed m-primary ideals (but we do not need this result for the case where the $\partial f/\partial z_i$ generate an m-primary ideal). Choose an integrally closed m-primary ideal $\mathfrak{A} \supseteq I$ with $f \notin \mathfrak{A}$. Then we can map R to a discrete valuation ring V in such a way that the image f is not in $\mathfrak{A}V$ (and, hence, not in IV), and it follows that m maps into the maximal ideal of V. Note that V cannot be just a field here, for then f maps to 0. Replace V by its completion: we may assume that V is complete. Since we are in the equal characteristic 0 case, the image of $\mathbb{C}$ in V can be extended to a coefficient field. Thus, we may assume that $V = L[[x]]$, where $\mathbb{C} \subseteq L$ and m maps into (x).

Let $h : R \to L[[x]]$ be the map, and $h(z_i) = g_i(x)$, $1 \leq i \leq n$. Then f maps to $f\big(g_1(x), \dots, g_n(x)\big)$. The key point is that the chain rule holds here, by a formal calculation. Thus,

$$\frac{d}{dx}\big(h(f)\big) = \sum_{i=1}^{n} h(\partial f/\partial z_i)\frac{dg_i(x)}{dx}.$$

It follows that the derivative of $h(f)$ is in IV. But over a field of characteristic 0, the derivative of a nonzero non-unit v has order exactly one less than that of v. Hence, $f \in IV$ as well. $\square$

Theorem (corollary of the Briançon-Skoda theorem). *With hypotheses as in the preceding Theorem, f^n is in the ideal generated by its partial derivatives.*

Proof. This is immediate from the preceding Theorem and the Briançon-Skoda Theorem. $\square$