Math 711: Lecture of September 25, 2006

Following Lipman and Sathaye [J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda*, Michigan Math. J. **28** (1981), 199–222] we present the Briançon-Skoda Theorem in a generalized form:

Briançon-Skoda Theorem (Lipman-Sathaye version). *Let R be a Noetherian normal domain, and let I_0 be an ideal of R such that $\mathrm{gr}_{I_0}(R)$ is regular. Let $n \geq 1$ and let I be an ideal of R generated over I by n elements, say $f_1, \ldots, f_n$. Let $k \geq 1$ be any positive integer. Then $\overline{I^{n+k-1}} \subseteq I^k$.*

The version stated in the Lecture of September 6 is the case where $I_0 = 0$ and $k = 1$. Notice that the result is non-trivial even when $n = 1$, where it states that all the powers of I are integrally closed.

We shall first explain how this result follows from the Lipman-Sathaye Jacobian Theorem (although this will take a while), and then focus on the proof of the latter. We need an intermediate result:

Theorem (Lipman-Sathaye). *Let B be a Noetherian normal integral domain, and let $v \in B - \{0\}$ be such that B/vB is regular. Let t denote the inverse of v in the fraction field of B. Let $f_1, \ldots, f_n \in B$ and let $S = B[f_1t, \ldots, f_nt]$. Let S' be the integral closure of S in its field of fractions. Then $v^{n-1}S' \subseteq S$.*

We want to see that the second theorem implies the first. We need some preliminary facts.

Lemma. *Let be any ring, and I an ideal of R.*

(a) *The integral closure of the extended Rees ring $R[It, v]$ in $R[t, v]$ in degree k is $\overline{I^k}t^k$ (if $k \leq 0$, let $I^k = R$). That is, the integral closure in $R[t, v]$ is*

$$\cdots + Rv^k + \cdots + Rv^2 + Rv + R + \overline{I}t + \overline{I^2}t^2 + \cdots + \overline{I^m}t^m + \cdots .$$

If R is a normal domain, this is also the integral closure of R in its fraction field.

(b) *Suppose that R is a Noetherian domain and that $\mathrm{gr}_I R$ is an integral domain (or that its localization at every prime ideal is an integral domain). Then every power of I is integrally closed.*

Proof. (a) The integral closure in $R[IT, v]$ is $\mathbb{Z}$-graded. The result for nonnegative degrees is clear. In positive degree k, if rt^k is integral over $R[It, v]$ it satisfies a monic polynomial of degree d for some d, and the sum of the coefficients of t^{dk} must be 0. Just as in the case of $R[It]$, this yields an equation establishing the integral dependence of r on I^k. The final statement follows because when R is normal, so is $R[t, v]$, and so $R[t, v]$ must contain the normalization of $R[It, v]$.

(b) If $\overline{I^n}/I^n \neq 0$, we may preserve this while localizing. Since integral closure commutes with localization, we may assume that R is local. If I expands to the unit ideal, there is nothing to prove. Otherwise, $\mathrm{gr}_I(R)$ is $\mathbb{N}$-graded over the local ring R/I. This implies that it is a domain: this is left as an exercise in Problem Set #2. When $\mathrm{gr}_I(R)$ is a domain, we can define a valuation on R whose value on a nonzero element r is the unique nonnegative integer h such that $r \in I^h - I^{h+1}$. I^n is then the contraction of the nth power of the maximal ideal of a discrete valuation ring. Again, the details are left as an exercise in Problem Set #2. $\square$

Proof that the second theorem implies the Briançon-Skoda theorem. Let $B = R[I_0t, v]$. Then $B/vB \cong \mathrm{gr}_{I_0}(R)$ is regular, and B is normal by the Lemma above. Then $S = B[f_1t, \dots, f_nt] = R[It, v]$ is the extended Rees ring of I over R. It follows that in degree $n+k-1$, S' is $\overline{I^{n+k-1}}t^{n+k+1}$. The fact that $v^{n-1}S' \subseteq S$ implies that $v^{n-1}[S']_{n+k-1} \subseteq [S]_k = I^kt^k$, and so $\overline{I^{n+k-1}} \subseteq I^k$. $\square$

Until further notice, R denotes a Noetherian domain with fraction field $\mathcal{K}$, and S denotes an algebra essentially of finite type over R (i.e., a localization at some multiplicative system of a finitely generated R-algebra) such that S is torsion-free and *generically étale* over R, by which we mean that $\mathcal{L} = \mathcal{K} \otimes_R S$ is a finite product of finite separable algebraic field extensions of $\mathcal{K}$. Note that $\mathcal{L}$ may also be described as the total quotient ring of S. We shall denote by S' the integral closure of S in $\mathcal{L}$. We shall prove that S' is module-finite over S if R is regular (and more generally).

If A and B are subsets of $\mathcal{L}$ we denote by $A :_{\mathcal{L}} B$ the set $\{u \in \mathcal{L} : uB \subseteq A\}$. If C is a subring of $\mathcal{L}$ and A is a C-module, then $A :_{\mathcal{L}} B$ is also a C-module.

We shall write $\mathcal{J}_{S/R}$ for the Jacobian ideal of S over R. If S is a finitely generated R-algebra, so that we may think of S as $R[X_1, \dots, X_s]/(f_1, \dots, f_h)$, then $\mathcal{J}_{S/R}$ is the ideal of S generated by the images of the size s minors of the Jacobian matrix $(\partial f_j/\partial x_i)$ under the surjection $R[X] \to S$. This turns out to be independent of the presentation, as we shall show below. Moreover, if $u \in S$, then $\mathcal{J}_{S_u/R} = \mathcal{J}_{S/R}S_u$. From this one sees that when S is essentially of finite type over R and one defines $\mathcal{J}_{S/R}$ by choosing a finitely generated subalgebra S_0 of S such that $S = W^{-1}S_0$ for some multiplicative system W of S_0, if one takes $\mathcal{J}_{S/R}$ to be $\mathcal{J}_{S_0/R}S$, then $\mathcal{J}_{S/R}$ is independent of the choices made. We shall consider the definition in greater detail later. The result we aim to prove is:

Theorem (Lipman-Sathaye Jacobian theorem). *Let R be regular domain*[1] *with fraction field $\mathcal{K}$ and let S be an extension algebra essentially of finite type over R such that S is torsion-free and generically étale over R. Let $\mathcal{L} = \mathcal{K} \otimes_R S$ and let S' be the integral closure of S in $\mathcal{L}$. Then $S' :_{\mathcal{L}} \mathcal{J}_{S'/R} \subseteq S :_{\mathcal{L}} \mathcal{J}_{S/R}$.*

[1]We can weaken the regularity hypotheses on R quite a bit: instead, we may assume that R is a Cohen-Macaulay Noetherian normal domain, that the completion of every local ring of R is reduced, and that for every height one prime Q of S', if $P = Q \cap R$, then R_P is regular.

Note that, since $\mathcal{J}_{S'/R}$ is an ideal of S', we have that $S' \subseteq S' :_{\mathcal{L}} \mathcal{J}_{S'/r}$. The statement that $S' \subseteq S :_{\mathcal{L}} \mathcal{J}_{S/R}$ implies that $\mathcal{J}_{S/R}S' \subseteq S$, i.e., that $\mathcal{J}_{S/R}$ "captures" the integral closure S' of S (all we mean by this is that it multiplies S' into S).

We next want to explain why the Jacobian ideal is well-defined. We assume first that S is finitely presented over R. To establish independence of presentation we first show that this ideal is independent of the choice of generators for the ideal I. Obviously, it can only increase as we use more generators. By enlarging the set of generators still further we may assume that the new generators are obtained from the orginal ones by operations of two kinds: multiplying one of the original generators by an element of the ring, or adding two of the original generators together. Let us denote by ∇f the column vector consisting of the partial derivatives of f with respect to the variables. Since $\nabla(gf) = g\nabla f + f\nabla g$ and the image of a generator f in S is 0, it follows that the image of $\nabla(gf)$ in S is the same as the image of $g\nabla f$ when $f \in I$. Therefore, the minors formed using $\nabla(gf)$ as a column are multiples of corresponding minors using ∇f instead, once we take images in S. Since $\nabla(f_1 + f_2) = \nabla f_1 + \nabla f_2$, minors formed using $\nabla(f_1 + f_2)$ as a column are sums of minors from the original matrix. Thus, independence from the choice of generators of I follows.

Now consider two different sets of generators for S over R. We may compare the Jacobian ideals obtained from each with that obtained from their union. This, it suffices to check that the Jacobian ideal does not change when we enlarge the set of generators $f_1, \ldots, f_s$ of the algebra. By induction, it suffices to consider what happens when we increase the number of generators by one. If the new generator is $f = f_{s+1}$ then we may choose a polynomial $h \in R[X_1, \ldots, X_s]$ such that $f = h(f_1, \ldots, f_s)$, and if $g_1, \ldots, g_h$ are generators of the original ideal then $g_1, \ldots, g_h$, $X_{s+1} - h(X_1, \ldots, X_s)$ give generators of the new ideal. Both dimensions of the Jacobian matrix increase by one: the original matrix is in the upper left corner, and the new bottom row is $(0\ 0\ \ldots\ 0\ 1)$. The result is then immediate from

Lemma. *Consider an $h+1$ by $s+1$ matrix M over a ring S such that the last row is $(0\ 0\ \ldots\ 0\ u)$, where u is a unit of S. Let M_0 be the h by s matrix in the upper left corner of M, obtained by omitting the last row and the last column. Then $I_s(M_0) = I_{s+1}(M)$.*

Proof. If we expand a size $s+1$ minor with respect to its last column, we get an S-linear combination of size s minors of M_0. Therefore, $I_{s+1}(M) \subseteq I_s(M_0)$. To prove the other inclusion, consider any s by s submatrix Δ_0 of M_0. We get an $s+1$ by $s+1$ submatrix Δ of M by using as well the last row of M and the appropriate entries from the last column of M. If we calculate $\det(\Delta)$ by expanding with respect to the last row, we get, up to sign, $u\det(\Delta_0)$. This shows that $I_s(M_0) \subseteq I_{s+1}(M)$. $\square$

This completes the argument that the Jacobian ideal $\mathcal{J}_{S/R}$ is independent of the presentation of S over R.

We next want to observe what happens to the Jacobian ideal when we localize S at one (or, equivalently, at finitely many) elements. Consider what happens when we localize at $u \in S$, where u is the image of $h(X_1, \ldots, X_s) \in R[X_1, \ldots, X_s]$, where we have chosen an

R-algebra surjection $R[X_1, \dots, X_s] \twoheadrightarrow S$. We may use $1/u$ as an additional generator, and introduce a new variable X_{s+1} that maps to $1/u$. We only need one additional equation, $X_{s+1}h(X_1, \dots, X_s) - 1$, as a generator. The original Jacobian matrix is in the upper left corner of the new Jacobian matrix, and the new bottom row consists of all zeroes except for the last entry, which is $h(X_1, \dots, X_s)$. Since the image of this entry is u and so invertible in $S[u^{-1}]$, the Lemma above shows that the new Jacobian ideal is generated by the original Jacobian ideal. We have proved:

Proposition. *If S is a finitely presented R-algebra and T is a localization of S at one (or finitely many) elements, $\mathcal{J}_{T/R} = \mathcal{J}_{S/R}T$.* $\square$