Math 711: Lecture of September 27, 2006

We next want to extend the definition of the Jacobian ideal to the case where S is a localization of a finitely generated R-algebra, even though S itself may not be finitely generated over R.

Suppose that $S = W^{-1}S_1$ where S_1 is finitely generated over the Noetherian ring R. As mentioned earlier, we want to define $\mathcal{J}_{S/R}$ to be $\mathcal{J}_{S_1/R}S = W^{-1}\mathcal{J}_{S_1/R}$. We only need to check that the result is independent of the choice of S_1 and W. First note that we may replace S_1 by its image in S (and W by its image as well). To see this, let $\mathfrak{A}$ be the kernel of the map $S_1 \to S$. Then $\mathfrak{A}$ is killed by some element of $w \in W$, and, by the Proposition at the end of the Lecture Notes for September 25, we may replace S_1 by its localization at w. But $(S_1)_w$ injects into S. Let $T \cong S_1/\mathfrak{A}$ be the image of S_1 in S. Then $T_w \cong (S_1)_w$, and so $\mathcal{J}_{S_1/R}$ and $\mathcal{J}_{T/R}$ both expand to $\mathcal{J}_{T_w/R}$. It follows that $\mathcal{J}_{S_1/R}$ and $\mathcal{J}_{T/R}$ will have the same expansion to S. Now suppose that S is the localization of finitely generated R-subalgebras $S_i \subseteq S$ at the multiplicative systems W_i, $i = 1, 2$. We want to show that $J_{S_i/R}S$ is independent of i.

First note that each element in a finite set of generators for S_2 over R is multiplied into S_1 by an element of W_1. By multiplying these elements of W_1 together, we can find $w_1 \in W_1$ such that $S_2 \subseteq (S_1)_{w_1}$. Since S_1 and $(S_1)_{w_1}$ produce the same result when their Jacobian ideals are expanded to S, we may replace S_1 by $(S_1)_{w_1}$, and so assume that $S_2 \subseteq S_1$. But we can similarly find $w_2 \in W_2$ such that $S_1 \subseteq (S_2)_{w_2}$. Then $(S_1)_{w_2} = (S_2)_{w_2} = S_0$, say. It is then clear that $\mathcal{J}_{S_i/R}S = \mathcal{J}_{S_0}S$ for $i = 1, 2$. This shows that $\mathcal{J}_{S/R}$ is well-defined independent of choices.

There is another approach to defining $\mathcal{J}_{S/R}$ for localizations of finitely generated R-algebras. First note that given a derivation D of a ring T, i.e., a map $D : T \to T$ that is a homomorphism of additive groups and satisfying $D(f_1f_2) = f_1D(f_2) + D(f_1)f_2$ for all $f_1,\, f_2 \in T$, it induces a unique derivation $\widetilde{D} : W^{-1}T \to W^{-1}T$ such that diagram

$$\begin{array}{ccc} W_T & \xrightarrow{\widetilde{D}} & W_T \\ \uparrow & & \uparrow \\ T & \xrightarrow{D} & T \end{array}$$

commutes. One gets $\widetilde{D}$ by letting

$$\widetilde{D}(f/w) = \frac{wDf - fDw}{w^2}.$$

(One needs to check that this is well-defined.) In consequence the partial differentiation operators $\dfrac{\partial}{\partial X_j}$ extend uniquely from the polynomial ring $R[X_1,\, \ldots\, , X_s]$ to any localization

$W^{-1}R[X_1, \ldots, X_s]$. Given a finitely generated R-algebra S_0 and a multiplicative system $W_0 \subseteq S_0$, we can choose a surjection $T = R[X_1, \ldots, X_s] \twoheadrightarrow S_0$, and let W be the inverse image of W_0 in T, which is a multiplicative system in T. Then we have a surjection $W^{-1}T \twoheadrightarrow W_0^{-1}S_0 = S$, and so we can write $S \cong W^{-1}R[X_1, \ldots, X_s]/(f_1, \ldots, f_h)$. We can then define $\mathcal{J}_{S/R}$ as the expansion of $I_s\big((\partial f_j/\partial X_i)\big)$ to S. We leave it to the reader to show that this produces the same ideal as our earlier definition of $\mathcal{J}_{S/R}$

Remark. Suppose that we are calculating the Jacobian ideal of $S = R[X_1, \ldots, X_s]/I$ over R. If we modify the elements $f_1, \ldots, f_s \in I$ by adding elements $g_1, \ldots, g_s \in I^2$, the image of the Jacobian minor $\det(\partial f_j/\partial x_i)$ does not change. The point is that each of the partial derivatives of an element of I^2 is in I, by the product rule, and so the image of every partial derivative of any g_j in S is 0. We shall make use of this trick later.

Definition: the conductor. Let S be a reduced Noetherian ring and S' its integral closure. The *conductor*, denoted $\mathfrak{C}_{S'/S}$, for $S \subseteq S'$ is $\{s \in S : S's \subseteq S\}$.

It is easily checked that $\mathfrak{C}_{S'/S}$, which by definition is contained in S, is actually an ideal of S'. Thus, it is an ideal of both S and S'. It may also be characterized as the largest ideal of S which is an ideal of S'.

Examples. Let x be an indeterminate over the field K, and let $S = K[x^2, x^3] \subseteq K[x]$. Then $S' = K[x]$, and the conductor is the maximal ideal $(x^2, x^3)S$, which contains all powers of x. However if we let $T = K[x^3, x^5] \subseteq K[x]$, then T may also be described as $K[x^3, x^5, x^6, x^8, x^9, x^{10}, \ldots]$. It is still the case that $T' = K[x]$, but now the conductor is $(x^8, x^9, x^{10})T$.

We next state an easy Corollary of the Jacobian theorem (but keep in mind that we have not yet proved the Jacobian theorem).

Corollary. . *Let R be a regular local ring and let $f_1, \ldots, f_n$, $v_1, \ldots, v_n \in R$, with the $v_i \neq 0$. Let $S = R[f_1/v_1, \ldots, f_n/v_n]$. Then $v_1 \cdots v_n \in \mathcal{J}_{S/R}$ and, hence, $v_1 \cdots v_n S' \subseteq S$. In other words, $v_1 \cdots v_n \in \mathfrak{C}_{S/R}$.*

Proof. $S = R[X_1, \ldots, X_n]/I$ for an a suitable ideal I, where X_j maps to f_j/v_j. Hence, we can include the elements $v_jX_j - f_j$ as the first n generators of I, and it follows that the first n rows of the Jacobian matrix form a diagonal matrix with $v_1, \ldots, v_n$ on the diagonal. Hence, $v_1 \cdots v_n$ is one of the minors. $\square$

We want to restate the Jacobian theorem with a slight refinement that makes use of the basic facts about the conductor. The statement that $S' :_{\mathcal{L}} \mathcal{J}_{S'/R} \subseteq S :_{\mathcal{L}} \mathcal{J}_{S/R}$ is equivalent to the statement $\mathcal{J}_{S/R}(S' :_{\mathcal{L}} \mathcal{J}_{S'/R}) \subseteq S$. Since $S' :_{\mathcal{L}} \mathcal{J}_{S'/R}$ is an S'-module, so is the left hand side. Therefore, the left hand side is an ideal of S' that is contained in S, and so it is contained in $\mathfrak{C}_{S'/S}$. Therefore, we can reformulate the Jacobian theorem as follows:

Theorem (Lipman-Sathaye Jacobian theorem). *Let R be regular domain*[1] *with fraction field $\mathcal{K}$ and let S be an extension algebra essentially of finite type over R such that S is torsion-free and generically étale over R. Let $\mathcal{L} = \mathcal{K} \otimes_R S$ and let S' be the integral closure of S in $\mathcal{L}$. Then $\mathcal{J}_{S/R}(S' :_{\mathcal{L}} \mathcal{J}_{S'/R}) \subseteq \mathfrak{C}_{S'/S}$.*

We have already proved that the second Theorem of Lipman and Sathaye (stated on the first page of the Lecture Notes of September 25) implies the Lipman-Sathaye version of the Briançon-Skoda Theorem. The conclusion of this second Theorem can be phrased as follows: $v^{n-1} \in \mathfrak{C}_{S'/S}$. The easy Corollary of the Jacobian theorem stated above gives a *weakened* version of the result we want right away: we can take $v_1 = \cdots = v_n = v$, and we find that $v^n \in \mathfrak{C}_{S'/S}$ under the hypotheses of the second Theorem. This gives a likewise weakened version of the Briançon-Skoda theorem, in which the conclusion is that $\overline{I^{n+k}} \subseteq I^k$ for $k \geq 1$. We will need to do quite a bit of work to decrease the exponent on the left by one.

The Lemma we need to do this, whose proof will occupy us for a while, is the following:

Key Lemma. *Let S be essentially of finite type, torsion-free and genercially étale over R, a regular domain. Let $v \in R$ be such that R/vR is regular. Suppose $f \in R - vR$, and $f/v \in S$. Then $\mathcal{J}_{S'/R} \subseteq vS'$.*

The conclusion implies that $1/v \in S' :_{\mathcal{L}} \mathcal{J}_{S'/R}$. Coupled with the Jacobian Theorem, which tell us that $\mathcal{J}_{S/R}(S' :_{\mathcal{L}} \mathcal{J}_{S'/R}) \subseteq \mathfrak{I}$, we have that $\mathcal{J}_{S/R} \cdot (1/v) \subseteq \mathfrak{I}$, and since we already know that $v^n \in \mathcal{J}_{S/R}$, we can conclude that $v^n/v \in \mathfrak{I}$. Thus, $v^{n-1}S' \subseteq S$, as required.

Remark. For the moment we have been assuming that S' is module-finite over S: we shall prove this later.

There are two proofs that are still hanging: one is for the Key Lemma stated just above, and the other is for the Jacobian theorem itself. We shall address the Key Lemma first. We need several preliminaries.

Recall that the *embedding dimension* of a local ring $(T, \mathfrak{m}, L)$ is the least number of generators of $\mathfrak{m}$, and, by Nakayama's Lemma, is the same the L-vector space dimension of $\mathfrak{m}/\mathfrak{m}^2$. Also recall that a local ring is *regular* if its embedding dimension and Krull dimension are equal. By a theorem, a regular local ring is an integral domain. A minimal set of generators of $\mathfrak{m}$ is always a system of parameters and is called a *regular system of parameters*. If we kill one element in a regular system of parameters, the Krull dimension and embedding dimension both drop by one, and the ring is still regular. It follows that a regular system of parameters for a regular local ring T is a regular sequence, and that the quotient of T by the ideal generated by part of a regular system of parameters is again a regular local ring. The converse is true:

[1] Or: let R be a Cohen-Macaulay Noetherian normal domain such that the completion of every local ring of R is reduced, and such that for every height one prime Q of S', if $P = Q \cap R$, then R_P is regular.

Lemma. *Let* $(T, \mathfrak{m})$ *be a regular local ring and* $\mathfrak{A} \subseteq \mathfrak{m}$ *an ideal. Then* $S = T/\mathfrak{A}$ *is regular if and only if* $\mathfrak{A}$ *is generated by part of a regular system of parameters (equivalently, part of a minimal set of generators for* $\mathfrak{m}$*). In particular, if* T *and* S *are both regular than* $\mathfrak{A}$ *must be generated by* $\dim(T) - \dim(S)$ *elements.*

Proof. It remains only to show that if $T/\mathfrak{A}$ is regular, then $\mathfrak{A}$ is generated by part of a regular system of parameters. If $\mathfrak{A} = 0$ we may use the empty subset of a regular system of parameters as the set of generators. Assume $\mathfrak{A} \neq 0$. We use induction on $d = \dim(T)$. If $\mathfrak{A} \subseteq m^2$, then killing $\mathfrak{A}$ decreases the Krull dimension, but the embedding dimension stays the same. But then the quotient cannot be regular. Therefore, we may choose an element of x_1 of $\mathfrak{A}$ that is not in m^2. The element x_1 is part of a regular system of parameters. If $d = 1$ then x_1 generates $\mathfrak{m}$, which must be $\mathfrak{A}$, and we are done. If not, we have that T/x_1T is regular and we may apply the induction hypothesis to conclude $\mathfrak{A}/x_1T$ is generated by part of regular system of parameters $\overline{x}_2, \dots, \overline{x}_k$ for T/x_1T. Let x_j in T map to $\overline{x}_j$, $2 \leq j \leq d-1$. Then $(x_1, \dots, x_k) = T$ and $x_1, \dots, x_k$ is part of a regular system of parameters for T. $\square$

Notation. If we have m polynomials $F_1, \dots, F_m \in R[X_1, \dots, X_m]$, we write $\dfrac{\partial(F_1, \dots, F_m)}{\partial(X_1, \dots, X_m)}$ for $\det\big(\partial F_j/\partial X_i\big)$.

Also note that if T is a ring and W is a multiplicative system such that $W^{-1}T$ is quasilocal, then $W^{-1}T \cong T_{\mathcal{Q}}$, where $\mathcal{Q}$ is the contraction of the maximal ideal of $W^{-1}T$ to T. On the one hand, we have a map $T \to W^{-1}T$ such that the image of $T - \mathcal{Q}$ is invertible, and this induces a mpa $\mathcal{T}_{\mathcal{Q}} \to W^{-1}T$ that lifts the identity map $T \to T$. On the other hand, the map $T \to T_{\mathcal{Q}}$ must carry W outside $\mathcal{Q}T_{\mathcal{Q}}$, and so we get an induced map $W^{-1}T \to T_{\mathcal{Q}}$ that lifts the identity on T. It is then easy to check that these are mutual inverses.

Lemma. *Let* $(R, m, K) \subseteq (S, \mathfrak{m}, L)$ *be a local map of regular local rings such that* S *is essentially of finite type over* R *and the extension of fraction fields is algebraic. Then for some integer* m*,* S *is a localization of* $R[X_1, \dots, X_m]/(F_1, \dots, F_m)$ *for suitable polynomials* $F_1, \dots, F_m$*, and, hence,* $\mathcal{J}_{S/R}$ *is the principal ideal* $\dfrac{\partial(F_1, \dots, F_m)}{\partial(X_1, \dots, X_m)}S$*.*

Proof. Choose a surjection $R[X_1, \dots, X_m] \twoheadrightarrow S$, and let $\mathcal{Q}$ be the inverse image of the maximal ideal of S in $R[X_1, \dots, X_m]$. Thus, $S \cong R[X]_{\mathcal{Q}}/\mathfrak{A}$. We need to see that the number of generators of $\mathfrak{A}$ is m: we can assume that they are in $R[X]$ by clearing denominators if necessary, since elements of $R[X] - \mathcal{Q}$ are units. By the preceding Lemma, since both $R[X]_{\mathcal{Q}}$ and its quotient by $\mathfrak{A}$ are regular, $\mathfrak{A}$ is generated by height $\mathcal{Q} - \dim(S)$ elements, and we therefore want to show that height $\mathcal{Q} - \dim(S) = m$. To see this, first note that by the dimension formula,

$$(*) \quad \dim(S) = \dim(R) - \mathrm{tr.\,deg.}(L/K).$$

Since $\mathcal{Q} \supseteq m$, $\mathcal{Q}$ corresponds to a prime P of $K[X_1, \ldots, X_m]$, and

$$(**) \quad \text{height}\, \mathcal{Q} = \dim(R) + \text{height}\, P,$$

while L is the fraction field of $K[X_1, \ldots, X_m]/P$, and so

$$(***) \quad \text{tr.deg.}(L/K) = \dim(K[X_1, \ldots, X_m]/P) = m - \text{height}\, P.$$

Thus, using $(*)$ and $(**)$, we have:

$$\text{height}\, \mathcal{Q} - \dim(S) = \dim(R) + \text{height}\, P - \big(\dim(R) - \text{tr.deg.}(L/K)\big)$$

and using $(***)$ this is

$$\text{height}\, P + \text{tr.deg.}(L/K) = \text{height}\, P + (m - \text{height}\, P) = m,$$

as required. $\square$

Corollary. *Let $R_0 \subseteq R_1 \subseteq R_2$ be regular domains such that R_i is essentially of finite type over R_0 for $i = 1, 2$ and $\mathrm{frac}(R_2)$ is algeraic over $\mathrm{frac}(R_0)$. Then*

$$\mathcal{J}_{R_2/R_0} = \mathcal{J}_{R_2/R_1} \mathcal{J}_{R_1/R_0}.$$