**Math 711: Lecture of October 2, 2006**

**Remark.** The quadratic transform of $(R, m, K)$ along $V$ is independent of the choice of regular system of parameters. If $x \in m$ has minimum order in $V$, the quadratic transform is the localization of $R[m/x]$ at the contraction $P$ of the maximal ideal of $V$. If $y$ also has minimum order, then $y/x \notin P$, and so $x/y \in R[m/x]_P$. Since $(u/x)(x/y) = u/y$ for all $u \in m$, it follows that

$$R[m/y] \subseteq R[m/x]_P.$$

If $Q$ is the contraction of the maximal ideal of $V$ to $R[m/y]$, we have that elements of $R[m/y] - Q$ are not in $PR[m/x]_P$, and therefore have inverses in $R[m/x]_P$. Thus,

$$R[m/y]_Q \subseteq R[m/x]_P.$$

The other inclusion follows exactly similarly. $\square$

**Remark.** The first quadratic transform $(T_1, m_1, K_1)$ of the regular local ring $(R, m, K)$ has dimension at most $\dim(R)$, and $\dim(T) = \dim(R) - \mathrm{tr.deg.}(K_1/K)$. In fact, since $R$ is regular, it is universally catenary and the dimension formula holds. Since the two fraction fields are equal, the transcendence degree of the the extension of fraction fields is 0. $\square$

**Remark.** A local inclusion of valuation domains with the same fraction field must be the identity map. For say that $(W, m_W) \subseteq (V, m_V)$ with $m_W \subseteq m_V$. If $f \in V - W$, then $1/f \in W$. Since $f \notin W$, $1/f \in m_W$. But then $1/f \in m_V$, which contradicts $f \in V$. $\square$

**Lemma.** *Let $x_1, \ldots, x_d$ be a regular sequence in the ring $R$ with $d \geq 2$.*

(a) *Consider $T = R[x_2/x_1] \subseteq R_{x_1}$. Then $T \cong R[X]/(x_1X - x_2)$. Moreover, $x_1, x_3, \ldots .x_d$ is a regular sequence in $R[x_2/x_1]$.*

(b) *Consider $T = R[x_2/x_1, \ldots, x_d/x_1] \subseteq R_{x_1}$. Then*

$$T \cong R[X_2, \ldots, X_d]/(x_1X_i - x_i : 2 \leq i \leq d).$$

*Moreover, $\mathcal{J}_{T/R} = x_1^{d-1}T$.*

*Proof.* (a) Since $x_1, x_2$ is a regular sequence in $R$, $x_1, x_1X - x_2$ is a regular sequence in $R[X]$: killing $x_1$ produces $(R/x_1R)[X]$, and the image of the second element is $-x_2$.

We claim that $x_1$ is not a zerodivisor modulo $(x_1X - x_2)$, for if $x_1f = (x_1X - x_2)g$ in $R[X]$, then $g = hx_1$ by the paragraph above. Since $x_1$ is not a zerodivisor in $R[X]$, we find that $f = (x_1X - x_2)h$.

This means that $R[X]/(x_1X - x_2)$ injects into its localization at $x_1$, which we may view as $R_{x_1}/(x_1X - x_2)$. Since $x_1$ is a unit, we may take $X - x_2/x_1$ as a generator of the ideal in the denominator, and so the quotient is simply $R_{x_1}$. The image of $R[X]/(x_1X - x_2)$ is then $R[x_2/x_1] \subseteq R_{x_1}$.

The last statement in part (a) follows because if we kill $x_1$ in $R[X]/(x_1X - x_2)$, we simply get $\big(R/(x_1,\, x_2)\big)[X]$. The images of $x_3, \, \ldots \, , x_d$ form a regular sequence in this polynomial ring, because that was true in $R/(x_1,\, x_2)$.

Part (b) now follows by induction on $d$: as we successively adjoin $x_2/x_1$, $x_3/x_1$ and so forth, the hypothesis we need for the next fraction continues to hold. The final statement is then immediate, because the Jacobian matrix calculated from the given presentation is $x_1$ times the size $d-1$ identity matrix. $\square$

We also note:

**Proposition.** *Let $(R,\, m,\, K) \subseteq V$ be an inclusion of a regular local ring in a DVR, let $v \in m$ be such that $R/vR$ is regular, and let $x \in m$ have minimal order in $V$. Let $T$ be the first quadratic transform of $R$ along $V$. Then either $v_1 = v/x$ is a unit of $T$, or else $T/v_1T$ is regular. If the first quadratic transform is a DVR, it is always the case that $v_1$ is a unit.*

*Proof.* Extend $v$ to a regular system of parameters $\mathcal{S}$ for $R$. If $v$ itself has minimum order in $V$, then $v/x$ is a unit of $T$. If not, then $x$ is a unit of $T$ times some element $x_1 \in \mathcal{S}$, and $v_1T = v_1'T$ if $v_1' = v/x_1$. Hence, we may assume without loss of generality that $x = x_1$ and $v = x_2$ in the regular system of parameters $\mathcal{S}$. Then $x_1$ and $x_2/x_1$ are in the maximal ideal of $T$, and to show that they form a regular sequence in $T$, it suffices to show that they form a regular sequence in the ring

$$R[x_1,\, x_2/x_1,\, \ldots\, ,\, x_d/x_1].$$

However, mod $(x_1)$, this ring becomes the polynomial ring $K[\overline{x}_2,\, \ldots\, ,\, \overline{x}_d]$, and the image of $x_2/x_1$ is $\overline{x}_2$.The quotient of $T$ by this regular sequence is a localization of the polynomial ring $K[\overline{x}_3,\, \ldots\, ,\, \overline{x}_d]$, and so is regular. Hence, $x_1, x_2/x_1$ is part of a regular system of parameters for $T$, and so $x_2/x_1 = v/x = v_1$ is a regular parameter. Note that in this case $di(T) \geq 2$, so that if $T$ is a DVR we must have that $v_1$ is a unit. $\square$

We are now ready to prove the result mentioned at the end of the Lecture Notes for September 29 concerning finiteness of the sequence of quadratic transforms under certain conditions: we follow the treatment in [S. Abhyankar, *Ramification theoretic methods in algebraic geometry*, Annals of Mathematics Studies Number **43**, Princeton University Press, Princeton, New Jersey, 1959], Propositon 4.4, p. 77.

**Theorem (finiteness of the quadratic sequence).** *Let $(R,\, m,\, K) \subseteq (V,\, \mathfrak{n},\, L')$ be a local inclusion of a regular local ring $R$ of dimension $d$ with fraction field $\mathcal{K}$ in a discrete valuation ring. Suppose that* $\mathrm{tr.\,deg.}(L'/K) = d-1$. *Let $T_0 = R$, and let $(T_i,\, m_i, K_i)$ denote the $i$ th quadratic transform of $R$ along $V$, so that for each $i \geq 0$, $T_{i+1}$ is the first quadratic transform of $T_i$ along $V$. Then this sequence is finite, and terminates at* *some $T_h$ (which means precisely that $T_h$ has dimension 1). Moreover, for this value of $h$, $T_h = V \cap \mathcal{K}$, so that $V \cap \mathcal{K}$ is essentially of finite type over $R$, and the transcendence degree of the residue field of $V \cap \mathcal{K}$ over $K$ is $d-1$.*

*Proof.* Assume that the sequence is infinite. By the dimension formula, for every $i$,

$$\dim(T_i) = \dim(R) - \mathrm{tr.\,deg.}(K_i/K),$$

so that

$$\mathrm{tr.\,deg.}(L'/K_i) = \mathrm{tr.\,deg.}(L'/K) - \mathrm{tr.\,deg.}(K_i/K) =$$

$$d - 1 - \mathrm{tr.\,deg.}(K_i/K) = \bigl(\dim(R) - \mathrm{tr.\,deg.}(K_i/K)\bigr) - 1 = \dim(T_i) - 1.$$

Therefore, at every stage, we have that $T_i \subseteq V$ satsifies the same condition that $R \to V$ did. Since the dimension is non-increasing it is eventually stable, and, by replacing $R$ by $T_j$ for $j \gg 0$, we might as well assume that the dimension of $T_i$ is stable througout. We call the stable value $d$, and we may assume that $d \geq 2$. It follows that every $K_i$ is algebraic over $K$. We have a directed (in fact, non-decreasing) union $\bigcup_i T_i$ of local rings inside $V$: call the union $(W, N, L)$. Here, $N$ is the union of the $m_i$ and $L$ is the union of the $K_i$, and so is algebraic over $K$.

We claim that $W$ must be a valuation domain of $\mathcal{K}$. If not, choose $x \in \mathcal{K}$ such that neither $x$ nor $1/x$ is in $W$, i.e., neither is in any $T_i$. Write $x = y_0/z_0$ where $y_0$, $z_0 \in T_0 = R$. These are both in the maximal ideal of $T_0$ (if $z_0$ were a unit, we would have $x \in T_0$, while if $y_0$ were a unit, we would have $1/x \in T_0$). If $u_0$ is a minimal generator of $m_0$ of minimum order in $V$, then $x = y_1/z_1$ where $y_1 = y_0/u_0$ and $z_1 = z_0/u_0$ are in $T_1$. We once again see that $y_1$ and $z_1$ must both belong to $m_1$. These have positive order in $V$, but $\mathrm{ord}_V(y_0) > \mathrm{ord}_V(y_1)$ and $\mathrm{ord}_V(z_0) > \mathrm{ord}_V(z_1)$. We recursively construct $y_i$ and $z_i$ in $m_i$ such that $x = y_i/z_i$, and $\mathrm{ord}_V(y_0) > \cdots > \mathrm{ord}_V(y_i)$ while $\mathrm{ord}_V(z_0) > \cdots > \mathrm{ord}_V(z_i)$. At the recursive step let $u_i$ be a minimal generator of $m_i$ such that $\mathrm{ord}_V(u_i)$ is minimum. Then $x = y_{i+1}/z_{i+1}$ where $y_{i+1} = y_i/u_i \in T_{i+1}$ and $z_{i+1} = z_i/u_i \in T_{i+1}$. As before, the fact that $x \notin T_{i+1}$ and $1/x \notin T_{i+1}$ yields that $y_{i+1}$ and $z_{i+1}$ are both in $m_{i+1}$, as required, and we also have that $\mathrm{ord}_V(y_i) > \mathrm{ord}_V(y_{i+1})$ and $\mathrm{ord}_V(z_i) > \mathrm{ord}_V(z_{i+1})$. This yields that $\{\mathrm{ord}_V(y_i)\}_i$ is a strictly decreasing sequence of nonnegative integers, a contradiction. It follows that $W$ is a valuation domain.

Since $W \subseteq V \cap \mathcal{K}$ is a local inclusion of valuation domains with the same fraction field, we can conclude from the third Remark on the first page of the Notes for this Lecture that $W = V \cap \mathcal{K}$. Therefore, $V \cap \mathcal{K}$ is essentially of finite type over $R = T_0$, and we have from the dimension formula that $\mathrm{tr.\,deg.}(L/K) = \dim(R) - \dim(W) = d - 1 \geq 1$, contradicting our earlier conclusion that $L$ is algebraic over $K$. This contradiction establishes the result. $\square$

*Proof of the Key Lemma, second step.* We recall the situation: $R$ is a regular domain with fraction field $\mathcal{K}$, $v$ is an element such that $R/vR$ is regular, $S$ is a reduced torsion-free algebra essentially of finite type over $R$ that is generically étale, $y \in R - vR$ is such that $y/v \in S$, and we want to prove that $\mathcal{J}_{S'/R} \subseteq vS'$. In the first step, we replaced $S'$ by its localization $(V, \mathfrak{n}, L')$ at a minimal prime of $vS'$ and $R$ by its localization at the contraction of that minimal prime. Thus, we have that $(R,\, m,\, K) \subseteq V$ is local, where $\dim(R) = d$. Since $V$ is essentially of finite type over $R$, the dimension formula yields that the $\operatorname{tr.deg.}(L'/K) = d - 1$. It will suffice to show that $\mathcal{J}_{V/R} \subseteq vV$. Note that if $d = 1$, then adjoining $y/v$ makes $v$ invertible in $V$, and there is nothing to prove. Thus, we may assume that $d \geq 2$. Consider the sequence of quadratic transforms

$$R = (T_0,\, m_0,\, K_0) \subseteq \cdots \subseteq (T_i,\, m_i,\, K_i) \subseteq \cdots \subseteq (T_h,\, m_h,\, K_h) \subseteq \cdots$$

By the Theorem on the finiteness of the quadratic sequence, we have that for some $h$, $T_h = V \cap \mathcal{K} = W$, which is therefore essentially of finite type over $R$. The condition $y/v \in S - R$ shows that $h \geq 1$. By the multiplicative property of Jacobian ideals stated in the Corollary at the end of the Lecture of September 27, we have that $\mathcal{J}_{V/R} = \mathcal{J}_{V/W}\mathcal{J}_{W/R}$. It therefore suffices to prove that $\mathcal{J}_{W/R} \subseteq vW$, and so we may henceforth assume that $V = W$ has fraction field $\mathcal{K}$ and is obtained from $R$ by a finite sequence of quadratic transforms along $V$.