Math 711: Lecture of October 4, 2006

Proof of the Key Lemma: final step. We are now in the case where $(R,\, m,\, K)$ is regular local of dimension d, $(R,\, m,\, K) \subseteq (V, \mathfrak{n}, L)$ is local, where $V \subseteq \mathcal{K}$, the fraction field of R, and V is a DVR essentially of finite type over R. In particular, $\mathrm{tr.\,deg.}(L/K) = d-1$ and we have a finite sequence of quadratic transforms along V

$$(R,\, m,\, K) = (T_0,\, m_0, K_0) \subseteq (T_1,\, m_1, K_1) \subseteq \cdots \subseteq (T_h,\, m_h,\, K_h) = (V,\, \mathfrak{n},\, L)$$

where $h \geq 1$, and $\dim(T_i) = d_i \geq 2$ for $i < h$. We must show that $\mathcal{J}_{V/R} \subseteq vV$, where $v \in R$ is such that R/vR is regular. Let $u_i \in m_i$ have minimum order in V for $0 \leq i \leq h-1$. Let $v_1 = v/u_0 \in T_1$. Recursively, so long as $v_i \in T_i$ is not a unit, we know by induction and the Proposition on p. 2 of the Lecture Notes of October 2 that v_i is a regular parameter in T_i and we may define $v_{i+1} = v_i/u_i \in T_{i+1}$. The Proposition just cited shows that either v_{i+1} is a unit of T_{i+1} or a regular parameter.

For some smallest $k \leq h$, v_k is a unit of T_k, for the same Proposition shows that if v_{h-1} is not a unit of T_{h-1}, then v_h is a unit of T_h: see the final statement of that same Proposition. Then

$$v = u_0 v_1 = u_0 u_1 v_2 = \cdots = u_0 u_1 \cdots u_{k-1} v_k,$$

where v_k is a unit of T_k and, hence, of V. Thus,

$$(*) \quad vV = u_0 u_1 \cdots u_{k-1} V$$

for some $k \leq h$. On the other hand, the Corollary at the end of the Lecture of September 27 shows that $\mathcal{J}_{V/R} = \mathcal{J}_{T_1/T_0} \mathcal{J}_{T_2/T_1} \cdots \mathcal{J}_{T_h/T_{h-1}}$. The last statement in part (b) of the Lemma on the first page of the Lecture Notes of October 2 shows that $\mathcal{J}_{T_{i+1}/T_i}$ is generated by $u_i^{d_i - 1}$. Thus,

$$(**) \quad \mathcal{J}_{V/R} = u_0^{d_0 - 1} \cdots u_{h-1}^{d_{h-1} - 1} V,$$

and each exponent $d_j - 1$ is at least one. The inclusion $\mathcal{J}_{V/R} \subseteq vV$ is now obvious from inspection of $(*)$ and $(**)$. $\square$

The module-finite property for normalizations.

We shall now address the problem of proving that S' is module-finite over S. Several of the details of the argument are left to the reader in Problem Set #2. The result we aim to prove is this:

Theorem (finiteness of the normalization). *Let S be torsion-free, generically étale, and essentially of finite type over a normal Noetherian domain R. Suppose that the completion of every local ring of R is reduced (which holds if R is either regular or excellent). Then the normalization S' of S over R is module-finite over S.*

We first give some preliminary results.

Lemma. *Let R be a Noetherian domain and b a nonzero element such that R_b is normal.*

(a) R is normal if and only if R_P is normal for every associated prime of b.

(b) $\{Q \in \operatorname{Spec}(R) : R_Q \text{ is not normal}\}$ is the union of the sets $V(P)$, where P is an associated prime of P such that R_P is not normal, and so is Zariski closed.

(c) If R_m has module-finite integral closure for every maximal ideal m of R, then R has module-finite integral closure.

We refer the reader to problem 4. in Problem Set #2.

Let $S = R[a_1/b_1, \ldots, a_h/b_h]$ where $b_1, \ldots, b_h$ are nonzerodivisors in R. This is a subring of R_b, where $b = b_1 \cdots b_h$, and each fraction can be written in the form a'_i/b. We may include b/b among these fractions, and so assume that some $a'_i = b$. Since every R-linear combination of the fractions is in S, if I is the ideal of R generated by the a'_i we have that $S = R[I/b]$, where $I/b = \{i/b : i \in I\} \subseteq R_b$. Here, I is an ideal containing b. We next observe:

Theorem (Rees). *Let (R, m, K) be a local ring such that $\widehat{R}$ is reduced. Let S be the ring obtained by adjoining finitely many fractions (elements of the total quotient ring) to R. Then the normalization S' of S is module-finite over S.*

Proof. If the ring itself is complete, the second problem from Problem Set #1 shows that one may reduce to the case where R is a complete local domain, and since we already know that the normalization of a complete local domain is normal, we may even assume that R is normal. We then want to prove that the normalization of $S = R[I/f]$ is module-finite over $R[I/f]$ with $f \in I$, by the discussion just above. This follows from the fifth and sixth problems of Problem Set #2.

Now consider the case where S itself is not necessarily complete. If the result fails, then there is an infinite strictly ascending chain $S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ of algebras generated by fractions over R, where S_{j+1} is module-finite over S_j for $j \geq 0$. But the chain $\widehat{R}[S_j]$ must be eventually stable, since the complete reduced ring $\widehat{R}$ will have finite normalization, so that $\widehat{R}[S_{j+1}] = \widehat{R}[S_j]$ for large j. But this implies $S_{j+1} = S_j$: otherwise some fraction over R is an $\widehat{R}$-linear combination of finitely many other such fractions but not an R-linear combination of them, and we can get a contradiction from this as follows. We may use a common denominator $r \in R$, not a zerodivisor, and write $a/r = \sum_{i=0}^{s}(a_i/r)\beta_i$ where the $a_i \in R$ and the $\beta_i \in \widehat{R}$. But then $a \in (a_1, \ldots, a_s)\widehat{R} \cap R$, which implies $a \in (a_1, \ldots, a_s)R$, say $a = \sum_{i=0}^{s} a_i b_i$ with $b_i \in R$, and that means we can replace the β_i by elements b_i of R. $\square$

We also need the fact that finite separable algebraic field extensions do not disturb the property of being reduced:

Lemma. *Let B denoted a reduced ring containing a field $\mathcal{K}$, and let $\mathcal{L}$ be a finite seprable algebraic extension of $\mathcal{K}$. Then $\mathcal{L} \otimes_{\mathcal{K}} B$ is reduced.*

Proof. B is the directed union of finitely generated $\mathcal{K}$-algebas B_0: since $\mathcal{L}$ is $\mathcal{K}$- flat, $\mathcal{L} \otimes_{\mathcal{K}} B$ is the directed union of the rings $\mathcal{L} \otimes_{\mathcal{K}} B_0$. Therefore, we may assume that B is finitely generated over $\mathcal{K}$, and has finitely many minimal primes. B embeds in its total quotient ring, which is a finite product of fields. Thus, we may replace B by its total quotient ring, it suffices to prove the result when B is a product fields. It is easy to see that it suffices to consider the case where B is a field, and we may even enlarge B further to an algebraically closed field Ω. But $\mathcal{L} \cong \mathcal{K}[x]/f(x)$ where f is monic with distinct roots in Ω, and so $\mathcal{L} \otimes_{\mathcal{K}} \Omega \cong \Omega[x]/\big(f(x)\big)$. This ring is isomorphic with a finite product of copies of Ω by the Chinese Remainder Theorem, since the roots of f are distinct. □

We are now ready to prove the main theorem stated earlier on the module-finite property for S' over S.

Proof of the finiteness of the normalizaion. We first replace S by a subring finitely generated over R of which it is a localization. Since localization commutes with normalization, it suffices to consider this subring. Thus, we may assume that S is finitely generated as an R-algebra. Second, the integral closure of S is the product of the integral closures of the domains obtained by killing a minimal prime of S. Thus, without loss of generality, it suffices to consider the case where S is a domain.

Each of the generators in a finite set of generators for S over R satisfies an algebraic equation over R with leading coefficient r_ν, say, and it follows that we may choose a nonzero element $r \in R$, the product of these leading coefficients, such that $S[1/r]$ is integral over $R[1/r]$. The integral closure of the normal domain $R[1/r]$ in the fraction field $\mathcal{L}$ of $S[1/r]$ is the same as the normalization of $S[1/r]$, and is module-finite over $S[1/r]$ by the first Theorem on p. 3 of the Lecture Notes of September 11. It follows that we may enlarge S by adjoining finitely many elements of its normalization and so obtain a domain with the property that $S[1/r]$ is normal for some nonzero r. It then follows from the first Lemma above that in order to prove that S has finite normalization, it suffices to prove this for S_Q for every maximal ideal Q of S. Choose $s \in S$ such that it generates $\mathcal{L}$ over $\mathcal{K}$. Let P be the contraction of Q to $R_1 = R[s]$. Then S_Q is a localization of $(R_1)_P[S]$, and S is generated over $(R_1)_P$ by elements of its fraction field.

Thus, to finish the argument, it suffices to show that the completion of $(R_1)_P$ is reduced. We may replace R by its localization at the contraction of P, and so we may assume that R is local with reduced completion. The completion of $(R_1)_P$ is one of the local rings of the completion of R_1 with respect to the maximal ideal of R. Thus, it suffices to show that this completion of R_1 is reduced. But this is $R_1 \otimes_R \widehat{R} \subseteq \mathcal{L} \otimes_R \widehat{R} \cong \mathcal{L} \otimes_{\mathcal{K}} (\mathcal{K} \otimes_R \widehat{R})$, and the result follows from the preceding Lemma because $\mathcal{K} \otimes_R \widehat{R}$ is reduced and $\mathcal{L}$ is finite separable algebraic over $\mathcal{K}$. □

Sketch of the proof of the Jacobian theorem.

We are ready to tackle the proof of the Jacobian theorem, but we first sketch the main ideas of the argument and then fill in the details.

Step 1: The local case suffices. Note that it is enough to prove the result when S is replaced by its various localizations at maximal ideals. Thus, we may assume that S is local, although we shall only make this assumption at certain points in the proof. When S is local we may also replace R its localization at the contraction of the maximal ideal of S, and so there is likewise no loss of generality in assuming that R is local and that $R \to S$ is local homomorphism (i.e., the maximal ideal of R maps into that of S).

Step 2: presenting S over R. Let T denote a localization of $R[X_1, \dots, X_n]$ that maps onto S, and let I denote the kernel. Let U denote the complement in T of the set of minimal primes $P_1, \dots, P_r$ of of I in T. Since S is reduced, $I = \bigcap_{i=1}^{r} P_i$. Since S is a torsion-free R-module, the minimal primes of I do not meet R, and correspond to the minimal primes of $I(\mathcal{K} \otimes T)$. Since killing any of these minimal primes produces an algebraic extension of $\mathcal{K}$, they must correspond to maximal ideals of $\mathcal{K}[X_1, \dots, X_n]$, and it follows that the P_i all have the same height, which must be the same as the number of variables, n. Thus, $U^{-1}T$ is a semilocal regular ring in which each of the maximal ideals $\mathcal{M}_i = P_i U^{-1}T$ is generated by n elements.

Step 3: special sequences and the modules $W_{S/R}$. Call a sequence $g_1, \dots, g_n$ of n elements of I *special* if it generates each of the $\mathcal{M}_i$ and is a regular sequence in T. We shall show that special sequences exist, and that there are sufficiently many of them that the images of the elements $\det(\partial g_j/\partial X_i)$ in S such that $g_1, \dots, g_n$ is special generate the Jacobian ideal. Moreover, when $g_1, \dots, g_n$ is special the image of $\det(\partial g_j/\partial X_i)$ in S is not a zerodivisor in S, and so has an inverse in $\mathcal{L}$. Given $\theta : T \to S$ and a special sequence $g_1, \dots, g_n$ we define a map

$$\Phi\colon \frac{(g_1, \dots, g_n)T :_T I}{(g_1, \dots, g_n)T} \to \mathcal{L}$$

by sending the class of u to $\overline{u}/\gamma$, where $\overline{u}$ is the image of u in S and γ is the image of $\det(\partial g_j/\partial X_i)$ in S. It is clear that I kills $(g_1, \dots, g_n)T :_T I/(g_1, \dots, g_n)T$, so that this is an S-module. We shall show that Φ is injective. *A priori*, its image depends on the choice of $T \to S$ and on the choice of the special sequence $g_1, \dots, g_n$, but the image turns out to be independent of these choices. Therefore, once we have shown all this we will have constructed a finitely generated canonically determined S-module $W_{S/R} \subseteq \mathcal{L}$.

Step 4: the behavior of the $W_{S/R}$ and the main idea of the argument. It will turn out that, quite generally, $W_{S/R} \subseteq S :_{\mathcal{L}} \mathcal{J}_{S/R}$. Here, one should think of S as varying. The Jacobian Theorem then follows from two further observations. The first is that when S is normal, this is an equality The second is that when one enlarges S to $S_1 = S[s_1]$ by adjoining one integral fraction $s_1 \in \mathcal{L}$ (so that $S \subseteq S_1 \subseteq S'$), then $W_{S_1/R} \subseteq W_{S/R}$. Repeated application of this fact yields that $W_{S'/R} \subseteq W_{S/R}$ and then we have

$$S' :_{\mathcal{L}} \mathcal{J}_{S'/R} = W_{S'/R} \subseteq W_{S/R} \subseteq S :_{\mathcal{L}} \mathcal{J}_{S/R},$$

and we are done. In the sequel we shall systematically fill in the details of the argument just outlined.