Math 711: Lecture of October 6, 2006

Detailed proof of the Jacobian theorem: existence of sufficiently many special sequences.

Note first that if R itself is a field then $S = \mathcal{L}$ and $S' = S$, so that $\mathcal{J}_{S/R} = \mathcal{J}_{S'/R}$ and there is nothing to prove. If R is finite then R must be a field, since R is a domain, and therefore we may assume without loss of generality that R is infinite in the remainder of the proof.

We shall need prime avoidance in the following form (cf. [I. Kaplansky, *Commutative Rings*, Revised Edition, Univ. of Chicag Press, Chicago, 1974], Theorem 124, p. 90.):

Lemma (prime avoidance for cosets). *Let R be any commutative ring, $x \in R$, $I \subseteq R$ an ideal and $P_1, \ldots, P_k$ prime ideals of R. Suppose that the coset $x + I$ is contained in $\bigcup_{i=1}^k P_i$. Then there exists j such that $Rx + I \subseteq P_j$.*

Proof. If $k = 1$ the result is clear. Choose $k \geq 2$ minimum giving a counterexample. Then no two P_i are comparable, and $x + I$ is not contained in the union of any $k - 1$ of the P_i. Now $x = x + 0 \in x + I$, and so x is in at least one of the P_j: say $x \in P_k$. If $I \subseteq P_k$, then $Rx + I \subseteq P_k$ and we are done. If not, choose $i_0 \in I - P_k$. We can also choose $i \in I$ such that $x + i \notin \bigcup_{j=1}^{k-1} P_i$. Choose $u_j \in P_j - P_k$ for $j < k$, and let u be the product of the u_j. Then $ui_0 \in I - P_k$, but is in P_j for $j < k$. It follows that $x + (i + ui_0) \in x + I$, but is not in any P_j, $1 \leq j \leq k$, a contradiction. $\square$

The following somewhat technical "general position" lemma is needed to prove that Jacobian determinants arising from special sequences generate the Jacobian ideal.

Lemma (general position for generators). *Let $R \subseteq T$ be a commutative rings such that R is an infinite integral domain and let $P_1, \ldots, P_r$ be mutually incomparable prime ideals of T contracting to (0) in R. Let $N \geq n \geq 1$ be integers and let $M = (g_1 \ \ldots \ g_N)$ be a $1 \times N$ matrix over T with entries in $I = \bigcap_{j=1}^r P_r$. Let κ_j denote the field $T_{P_j}/P_jT_{P_j}$ for $1 \leq j \leq r$ and let V_j denote the κ_j-vector space $P_jT_{P_j}/P_j^2T_{P_j}$. Suppose that for all j, $1 \leq j \leq r$, the κ_j-span of the images of the g_t under the obvious map $I \subseteq P_j \to P_jT_{P_j} \twoheadrightarrow V_j$ has κ_j-vector space dimension at least n.*

Then one may perform elementary column operations on the matrix M over T so as to produce a matrix with the property that, for all j, $1 \leq j \leq r$, the images of any n of its distinct entries are κ_j-linearly independent elements of V_j.

Of course, the entries of the new matrix generate the ideal $(g_1, \ldots, g_N)T$.

Proof. First note that the infinite domain R is contained in each of the κ_j.

We proceed by induction on the number of primes. If there are no primes there is nothing to prove. Now suppose that $1 \le h \le r$ and that column operations have already been performed so that any n entries have κ_j-independent images in V_j if $j < h$. (If $h = 1$ we may use M as is, since no condition is imposed.) We need to show that we can perform elementary column operations so that the condition also holds for $j = h$. Some n of the entries have κ_h-independent images in V_h: by renumbering we may assume that these are $g_1, \dots, g_n$. We now show that by induction on a, $n+1 \le a \le N$ that we may perform elementary column operations on the matrix so that

(1) The images of the entries of the matrix in each V_j for $j < h$ do not change and

(2) Any n of the images of $g_1, \dots, g_a$ in V_h are independent.

Choose $t \in T$ so that it is in the primes P_j for $j < h$ but not in P_h. Thus, t has nonzero image τ in κ_h. Let v_j denote the image of g_j in V_h. We may assume that the images of any n of the elements $g_1, \dots, g_{a-1}$ are independent in V_h. Thus, it will suffice to show that there exist $r_1, \dots, r_n \in R$ such that the image of $g_a + tr_1g_1 + \cdots tr_ng_n$ is independent of any $n-1$ of the vectors $v_1, \dots, v_{a-1}$ in V_h, i.e., such that $v_a + \tau r_1 v_1 + \cdots \tau r_n v_n$ is independent of any $n-1$ of the vectors $v_1, \dots, v_{a-1}$. (Note that condition (1) is satisfied automatically because the image of t is 0 in each κ_j for $j < h$.)

For each set D of $n-1$ vectors in $v_1, \dots, v_{a-1}$, there is a nonzero polynomial f_D in n variables over κ_h, and whose nonvanishing at the point $(r_1, \dots, r_n)$ guarantees the independence of $v_a + \tau r_1 v_1 + \cdots \tau r_n v_n$ from the vectors in D. To see this, choose a κ_h-basis for the space spanned by all the v_j and write the vectors in D and $v_a + \tau X_1 v_1 + \cdots \tau X_n v_n$ in terms of this basis. Form a matrix C from the coefficients. We can choose values of the X_i in R that achieve the required independence, and this means that some $n \times n$ minor of C does not vanish identically. (If v_a is independent of the vectors in D take all the X_i to be zero. Otherwise, v_a is in the κ_h-span of D, while at least one of the n independent vectors $v_1, \dots, v_n$ is not, say v_ν, and we can take all the X_i except X_ν to be 0 and $X_\nu = 1$.) This minor gives the polynomial $f_D \in \kappa_h[X_1, \dots, X_n]$.

Choose a field extension $\mathcal{F}$ of $\mathcal{K} = \mathrm{frac}\,(R)$ that contains isomorphic copies of all of the κ_j. The product f of the f_D in $\mathcal{F}[x_1, \dots, x_n]$ as D varies through the $n-1$ element subsets of $v_1, \dots, v_{a-1}$ is then a nonzero polynomial in $\mathcal{F}[X_1, \dots, X_n]$, and so cannot vanish identically on the infinite domain R. Choose $r_1, \dots, r_n \in R$ so that $f(r_1, \dots, r_n) \ne 0$. Then every $f_D(r_1, \dots, r_n) \ne 0$ $\square$

Lemma. *Let $g_1, \dots, g_n$ be elements of a Noetherian ring T and let J be an ideal of T of depth at least n such that $(g_1, \dots, g_n)T + J$ is a proper ideal of T. If $g_1, \dots, g_i$ is a regular sequence in T (i may be zero, i.e., we may be assuming nothing about $g_1, \dots, g_n$) then there are elements $j_{i+1}, \dots, j_n \in J$ such that*

$$g_1, \dots, g_i, g_{i+1} + j_{i+1}, \dots, g_n + j_n$$

is a regular sequence in T.

In particular, there are elements $j_1, \dots, j_n \in J$ such that $g_1 + j_1, \dots, g_n + j_n$ is a regular sequence.

Proof. The last sentence is the case $i = 0$. We proceed by induction on $n - i$. If $i = n$ there is nothing to prove. We may pass to $T/(g_1, \ldots, g_i)T$, and so reduce to the case where $i = 0$. The image J is the same as the image of $J' = J + (g_1, \ldots, g_i)$ modulo $(g_1, \ldots, g_i)$. J' is a proper ideal of depth at least n, and so killing a regular sequence of length i in J' poduces an ideal of depth at least $n - i > 0$. Thus, we may assume that $i = 0$.

It then suffices to choose $j = j_1$ such that $g_1 + j$ is not a zerodivisor, for we may apply the induction hypothesis to construct the rest of the sequence. But if this were not possible we would have that $g_1 + j$ is contained in the union of the associated primes of (0) in T, and this implies that J is contained in an associated prime of (0) in T by the Lemma on prime avoidance for cosets proved at the beginning of this Lecture. This is a contradiction, since the depth of J is positive. $\square$

Theorem (existence of sufficiently many special sequences). *Let R be an infinite Cohen-Macaulay Noetherian domain and let S be a torsion-free generically étale R-algebra essentially of finite type over R. Let T be a localization of a polynomial ring in n variables over R that maps onto S, and let I be the kernel. Let $P_1, \ldots, P_r$ be the minimal primes of I in T. Then the Jacobian ideal $\mathcal{J}_{S/R}$ is generated by the images of elements* $\det(\partial g_j/\partial x_i)$ *such that $g_1, \ldots, g_n$ is a special sequence of elements of I, i.e., a regular sequence in I such that for every j, $1 \le j \le r$, $P_j T_{P_j} = (g_1, \ldots, g_n)T_{P_j}$.*

Proof. First choose generators $g_1, \ldots, g_N$ for I. Think of these generators as forming the entries of a $1 \times N$ matrix as in the Lemma on general position for generators. Each T_{P_j} is regular local of dimension n, so that each $P_j T_{P_j}/P_j^2 T_{P_j}$ has dimension n. It follows from the Lemma cited that we may assume without loss of generality that every n element subset of the generators $g_1, \ldots, g_N$ generates every $P_j T_{P_j}$. We know that the size n minors of the $n \times N$ matrix $(\partial g_j/\partial x_i)$ generate $\mathcal{J}_{S/R}$. Fix one of these minors: by renumbering, we may assume that it corresponds to the first n columns. It will suffice to show that the image of this minor in S is the same as the image of a minor coming from a special sequence. We may apply the preceding Lemma to choose elements $h_1, \ldots, h_n \in J = I^2$ such that $g_1 + h_1, \ldots, g_n + h_n$ is a regular sequence. This sequence is special: since $J = I^2 \subseteq P_j^2$ for all j, the elements generate each $P_j T_{P_j}$, and it was chosen to be a regular sequence. Finally, by the Remark near the top of p. 2 of the Lecture Notes of September 27, the image of the Jacobian determinant of $g_1 + h_1, \ldots, g_n + h_n$ in S is the same as the image of the Jacobian determinant of $g_1, \ldots, g_n$, and the result follows. $\square$