Math 711: Lecture of October 9, 2006

Lemma (comparison of special sequences). *Let R be an infinite Cohen-Macaulay Noetherian domain and let S be a torsion-free generically étale R-algebra essentially of finite type over R. Let T be a localization of a polynomial ring in n variables over R that maps onto S, and let I be the kernel. Let $P_1, \ldots, P_r$ be the minimal primes of I in T. Assume, moreover, that S and T are local. Let $\underline{g} = g_1, \ldots, g_n$ and $\underline{h} = h_1, \ldots, h_n$ be two special sequences in I. Then there is a finite chain of special sequences joining $\underline{g}$ to $\underline{h}$ such that for any two consecutive special sequences occurring in this chain, the sequences of elements occuring differ in at most one spot.*

Proof. We first show that given two special sequences $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ and an integer i, $1 \le i \le n$, we can choose $u \in T$ such that $g_1, \ldots, g_n$ remains special when g_i is replaced by u and $h_1, \ldots, h_n$ remains special as well when h_i is replaced by u. Since regular sequences (and, hence, special sequences) are permutable in a local ring, we may assume without loss of generality that $i = n$. Our first objective is to choose u such that, for all j, both sequences generate each $P_jT_{P_j}$. Since $P_j \cap R = (0)$ for every j, we have for each j that $\mathcal{K} \subseteq T_{P_j}$. To solve the problem we shall first choose u with the required property in $\mathcal{K} \otimes_R T$. For every j let C_j denote the contraction of

$$\big((g_1, \ldots, g_{n-1}) + P_j^2\big)T_{P_j}$$

to $\mathcal{K} \otimes_R T$, and let D_j denote the contraction of

$$\big((h_1, \ldots, h_{n-1}) + P_j^2\big)T_{P_j}$$

to $\mathcal{K} \otimes_R T$. The C_j and D_j together constitute finally many vector spaces over the field $\mathcal{K}$. We claim that they do not cover $\mathcal{K} \otimes_R I \subseteq \mathcal{K} \otimes_R T$, for if they did then one of them would contain $\mathcal{K} \otimes_R I$ (see the first Proposition on p. 3 of the Lecture Notes of September 18), and this would contradict the existence of g_n if it were one of the C_j, or the existence of h_n if it were one of the D_j. Hence, we can choose $u \in \mathcal{K} \otimes_R I$ with the required property. After multiplying by a suitable element of $R - \{0\}$, we may assume that u is in I, and it will still have the required property, since the multiplier is a unit in every R_{P_j}. Finally, as in the proof of the Theorem on existence of sufficiently many special sequences (which is the last Theorem of the Lecture of October 6), we can choose $v \in I^2$ such that $u + v$ is not a zerodivisor modulo either $(g_1, \ldots, g_{n-1})T$ nor modulo $(h_1, \ldots, h_{n-1})T$: this comes down to the assertion that $u + I^2$ is not contained in the union of the associated primes of the two ideals, and by the Lemma on prime avoidance for cosets, it suffices to show that I^2 is not contained in any of them. But this is clear because all the associated primes have height $n - 1$ while I^2 has height n.

Finally, to prove the existence of the chain of special sequences we use induction on the number of terms in which the two sequences, counting from the beginning, agree. Suppose that $g_i = h_i$ for $i < j$ while sequences $g_j \neq h_j$ (j may be 0 here). Then by the result of the paragraph above we may choose u such that the sequences

$$g_1, \, \ldots \, , \, g_{j-1}, u, g_{j+1}, \, \ldots \, , \, g_n$$

and

$$g_1, \, \ldots \, , \, g_{j-1}, u, h_{j+1}, \, \ldots \, , \, h_n$$

are both special. The first differs from $g_1, \, \ldots \, , g_n$ in only the j th spot, and the second differs from $h_1, \, \ldots \, , h_n$ in only the j th spot as well. By the induction hypothesis there is a chain of the required form joining these two, and the result follows. $\square$

The map Φ and the modules $W_{S/R}$.

Our next main goal is to construct the map Φ mentioned briefly in Step 3 of our Sketch of the proof the Jacobian theorem: see p. 4 of the Lecture Notes of October 4.

We first need a lemma whose proof involves universal modules of differentials or Kähler differentials.

In the next seven paragraphs, we assume only that $\mathcal{K}$ is a commutative ring and that T is $\mathcal{K}$-algebra. A $\mathcal{K}$-*derivation* of T into a T-module M is a map $D : T \to M$ such that

(1) D is a homomorphism of abelian groups

(2) For all $t_1, t_2 \in T$, $D(t_1 t_2) = t_1 D(t_2) + t_2 D(t_1)$ and

(3) For all $c \in \mathcal{K}$, $D(c) = 0$.

Condition (2) implies that D is $\mathcal{K}$-linear, for $D(ct) = cD(t) + tD(c) = cD(t) + t(0) = cD(t)$. Note that $D(1 \cdot 1) = 1D(1) + 1D(1)$, i.e. $D(1) = D(1) + D(1)$, so that condition (2) implies that $D(1) = 0$. In the presence of the other conditions, (2) is equivalent to $\mathcal{K}$-linearity, for if the map is $\mathcal{K}$-linear then for all $c \in \mathcal{K}$, $D(c \cdot 1) = cD(1) = c(0) = 0$.

There is a *universal $\mathcal{K}$-derivation* from a given $\mathcal{K}$-algebra T into a specially constructed T-module $\Omega_{T/\mathcal{K}}$. This is obtained as follows. Let G denote the T-free module whose basis consists of elements b_t in bijective correspondence with the elements t of T. Consider the T-submodule H of G spanned by elements of the three forms:

(1) $b_{t_1+t_2} - b_{t_1} - b_{t_2}$ for all t_1, $t_2 \in T$;

(2) $b_{t_1 t_2} - t_1 b_{t_2} - t_2 b_{t_1}$ for all t_1, $t_2 \in T$; and

(3) b_c for $c \in \mathcal{K}$.

We write $\Omega_{T/\mathcal{K}}$ for G/H. This is the *universal module of differentials* or the module of *Kähler differentials* for T over $\mathcal{K}$. Note that we have a map $d : T \to \Omega_{T/\mathcal{K}}$ that sends $t \mapsto b_t + H$, the class of b_t in G/H. The choice of elements that we killed (we took them as generators of H) precisely guarantees that d is a $\mathcal{K}$-derivation, the *universal $\mathcal{K}$-derivation* on T. It has the following universal property: given any T-module N and a $\mathcal{K}$-derivation $D : T \to N$, there is a unique T-linear map $L : M \to N$ such that $D = L \circ d$. To get the map L on G/H we define it on G so that the value on b_t is $D(t)$: this is forced if we are going to have $D = L \circ d$. One may check easily that H is killed, precisely because D is a $\mathcal{K}$-derivation, and this gives the required map. (It is also easy to see that the composition of d with any T-linear map *is* a $\mathcal{K}$-derivation.) Thus, for every T-module N, there is a bijection between the $\mathcal{K}$-derivatons of T into N and $\mathrm{Hom}_T(\Omega_{T/\mathcal{K}}, N)$.

Given generators t_i for T over $\mathcal{K}$, the elements dt_i (the index set may be infinite) span $\Omega_{T/\mathcal{K}}$. The value of d on a product of these generators is expressible in terms of the dt_i by iterated use of the product rule.

Given a polynomial ring $\mathcal{K}[X_i : i \in I]$ over $\mathcal{K}$, $\Omega_{T/\mathcal{K}}$ is the free T-module on the dX_i, and the universal derivation d is defined by the rule

$$dF = \sum_i \frac{\partial F}{\partial x_i} dX_i.$$

This formula is a consequence of the use of the iterated product rule, and it is straightforward to check that it really does give a derivation.

Note that if U is a multiplicative system in T, we have that $\Omega_{U^{-1}T/\mathcal{K}} \cong U^{-1}\Omega_{T/\mathcal{K}}$. Also observe that a $\mathcal{K}$-derivation $D : T \to M$ extends uniquely, via the rule $t/u \mapsto (uDt - tDu)/u^2$, to a $\mathcal{K}$-derivation $U^{-1}D : U^{-1}T \to U^{-1}M$ so that the diagram

$$\begin{array}{ccc} U^{-1}T & \xrightarrow{U^{-1}D} & U^{-1}M \\ \uparrow & & \uparrow \\ T & \xrightarrow{D} & M \end{array}$$

commutes.

The notations in the following Lemma are slightly different from those in our general setup.

Lemma. *Let $\mathcal{N}$ be a maximal ideal of $T = \mathcal{K}[X_1, \dots, X_n]$, a polynomial ring over a field $\mathcal{K}$. Assume that $\mathcal{L} = \mathcal{K}[X_1, \dots, X_n]/\mathcal{N}$ is separable field extension of $\mathcal{K}$. Then $g_1, \dots, g_n \in \mathcal{N}T_{\mathcal{N}}$ generate $\mathcal{N}T_{\mathcal{N}}$ if and only if the image of $\det\big(\partial g_j/\partial x_i\big)$ in $\mathcal{L}$ is not 0.*

Proof. Consider the universal $\mathcal{K}$-derivation $d : \mathcal{K}[X] \to \Omega_{\mathcal{K}[X]/\mathcal{K}}$, the module of Kähler differentials, which, as noted above, is the free T-module generated by the elements $dX_1, \dots, dX_n$. Of course, if $F \in \mathcal{K}[X]$ then $dF = \sum_{j=1}^n (\partial F/\partial x_j)\, dx_j$. The restriction of d to $\mathcal{N}$ gives a $\mathcal{K}$-linear map $\mathcal{N} \to \Omega_{\mathcal{K}[X]/\mathcal{K}}$, and, by the defining property of a derivation, it sends

$$\mathcal{N}^2 \to \mathcal{N}\Omega_{\mathcal{K}[X]/\mathcal{K}}.$$

Thus, there is an induced $\mathcal{K}$-linear map of $\mathcal{K}$-vector spaces

$$\delta : \mathcal{N}/\mathcal{N}^2 \to \mathcal{L} \otimes_T \Omega_{T/\mathcal{K}}.$$

Both modules are $\mathcal{L}$-vector spaces and it follows from the defining property of a derivation that δ is actually $\mathcal{L}$-linear (if $t \in T$ represents $\lambda \in \mathcal{L}$ and $u \in \mathcal{N}$, $d(tu) = tdu + udt$, and the second term will map to 0 in $\mathcal{L} \otimes_T \Omega_{T/\mathcal{K}}$). Since $T_{\mathcal{N}}$ is regular of dimension n, $\mathcal{N}/\mathcal{N}^2$ is an n-dimensional vector space over $\mathcal{L}$. The key point is that under the hypothesis that $\mathcal{L}$ is separable over $\mathcal{K}$, the map δ is an isomorphism of $\mathcal{L}$-vector spaces. To see this, observe that the map δ sends the elements represented by generators $g_1, \ldots, g_n$ for $\mathcal{N}T_{\mathcal{N}}$ to the elements represented by the dg_j, and so it has a matrix which is the image of the matrix $(\partial g_j/\partial x_i)$ after mapping the entries to $\mathcal{L}$. Thus, δ is an isomorphism if and only if the Jacobian determinant $\det(\partial g_j/\partial x_i)$ has nonzero image in $\mathcal{L}$. But this determinant generates $J_{\mathcal{L}/\mathcal{K}}$, and so δ is an isomorphism if and only if the Jacobian ideal of $\mathcal{L}$ over $\mathcal{K}$ is $\mathcal{L}$. But we may use any presentation of $\mathcal{L}$ over $\mathcal{K}$ to calculate $J_{\mathcal{L}/\mathcal{K}}$, and so we may instead use $\mathcal{L} \cong \mathcal{K}[Z]/\bigl(f(Z)\bigr)$ where Z here represents just one variable and where f is a single separable polynomial. The Jacobian determinant is then the value of $f'(Z)$ in $\mathcal{L}$, which is not zero by virtue of the separability.

Thus, δ is an $\mathcal{L}$-isomorphism. Moreover, we have already seen that if $g_1, \ldots, g_n$ are generators of $\mathcal{N}T_{\mathcal{N}}$ then the Jacobian determinant is not 0 in $\mathcal{L}$. But the converse is also clear, because if $g_1, \ldots, g_n$ are any elements of $\mathcal{N}T_{\mathcal{N}}$, they generate $\mathcal{N}T_{\mathcal{N}}$ if and only if their images in $\mathcal{N}T_{\mathcal{N}}/\mathcal{N}^2T_{\mathcal{N}} \cong \mathcal{N}/\mathcal{N}^2$ span this vector space over $\mathcal{L}$, by Nakayama's lemma, and this will be the case if and only if their further images in $\mathcal{L} \otimes_T \Omega_{T/\mathcal{K}}$ span that vector space over $\mathcal{L}$, since δ is an isomorphism. But that will be true if and only if the images of the dg_j span $\mathcal{L} \otimes_T \Omega_{T/\mathcal{K}}$, which is equivalent to the assertion that the images of the columns of the matrix $(\partial g_j/\partial x_i)$, after the entries are mapped to $\mathcal{L}$, span an n-dimensional space. This in turn is equivalent to the nonvanishing of $\det(\partial g_j/\partial x_i)$ in $\mathcal{L}$. $\square$

We now return to our standard set of notations and assumptions, as in Step 2 of the Sketch of the proof of the Jacobian theorem from p. 3 of the Lecture Notes of October 4. Thus, T is a localization of $R[X_1, \ldots, X_n]$ that maps onto S with kernel I. U is the complement in T of the set of minimal primes $P_1, \ldots, P_r$ of of I in T, and $I = \bigcap_{i=1}^r P_i$. The P_i do not meet R and correspond to the minimal primes of $I(\mathcal{K} \otimes T)$. The expansion of P_i to $U^{-1}T$ is maximal ideal $\mathcal{M}_i$ corresponding to a maximal ideal $\mathcal{N}_i$ of $\mathcal{K}[X_1, \ldots, X_n]$, and has height n. Here, $T_{P_i} \cong \mathcal{K}[X_1, \ldots, X_n]_{\mathcal{N}_i}$.

Corollary. *Let $g_1, \ldots, g_n \in I$. If $g_1, \ldots, g_n$ is a special sequence in I, then the image γ of $\det\bigl(\partial g_j/\partial X_i\bigr)$ is not a zerodivisor in S, and so represents an invertible element of the total quotient ring $\mathcal{L}$ of S.*

Proof. We may view $\mathcal{L}$ as the product of the fields $\mathcal{L}_i$, where $\mathcal{L}_i$ is the fraction field of T/P_i but may also be identified with $\mathcal{K}[X_1, \ldots, X_n]/\mathcal{N}_i$. It suffices to show that γ does not map to 0 under $\mathcal{L} \twoheadrightarrow \mathcal{L}_i$ for any i. The fact that the image of γ is not 0 in $\mathcal{L}_i$ follows from the preceding Lemma, the separability of $\mathcal{L}_i$ over $\mathcal{K}$, and the fact that for every i, $g_1, \ldots, g_n$ generates $P_iT_{P_i}$, which we may identify with $\mathcal{N}_i\mathcal{K}[X_1, \ldots, X_n]_{\mathcal{N}_i}$. $\square$

We continue the conventions in the paragraph preceding the statement of the Lemma, but because we shall let both S and its presentation vary we shall write θ for the map $T \to S$

and we shall denote by $\underline{g}$ a special sequence $g_1, \dots, g_n$ in I. We may then temporarily define

$$\Phi_{\theta,\underline{g}}\colon \frac{(g_1, \dots, g_n)T :_T I}{(g_1, \dots, g_n)T} \to \mathcal{L}$$

by sending the class of u to $\overline{u}/\gamma$ where $\overline{u}$ is the image of u in $\mathcal{L}$, and γ is the image of $\det(\partial g_j/\partial x_i)$ in $\mathcal{L}$: the element γ is invertible in $\mathcal{L}$ by the Corollary just above. The map is well defined because the g_i vanish under the map to $\mathcal{L}$. We shall often write Φ when θ and $\underline{g}$ are understood. We shall soon show that the image of Φ is contained in $S :_{\mathcal{L}} J_{S/R}$. Once this is established we shall change the definition of Φ very slightly by restricting its range to be $S :_{\mathcal{L}} J_{S/R} \subseteq \mathcal{L}$.

We note that

$$\frac{(g_1, \dots, g_n)T :_T I}{(g_1, \dots, g_n)T} \cong \operatorname{Hom}_T\big(T/I,\, T/(g_1, \dots, g_n)T\big).$$

We shall denote the image of $\Phi_{\theta,g}$ in $\mathcal{L}$ by $W_{S/R}(\theta, \underline{g})$. However, we shall see just below that it is independent of the choices of θ and $\underline{g}$, and once we know this we shall simply write it as $W_{S/R} \subseteq \mathcal{L}$.

Lemma. *With notation as above, $\Phi_{\theta,\underline{g}}$ is injective.*

Proof. Under the map $T \to \mathcal{L}$ the complement U of the union of the primes P_i becomes invertible. Because $\underline{g}$ is a regular sequence in T, every associated prime is minimal, and so no element of U is a zerodivisor on $T/(g_1, \dots, g_n)T$. Thus,

$$\frac{(g_1, \dots, g_n)T :_T I}{(g_1, \dots, g_n)T} \hookrightarrow \frac{T}{(g_1, \dots, g_n)T} \hookrightarrow \frac{U^{-1}T}{U^{-1}(g_1, \dots, g_n)T} \cong \mathcal{L}.$$

The map $\Phi_{\theta,\underline{g}}$ is the composition of this composite injection with multiplication by the invertible element $1/\gamma$ in $\mathcal{L}$. $\square$