Math 711: Lecture of October 11, 2006

We showed in the Lecture of October 9 that the map $\Phi_{\theta,\underline{g}}$ is injective. We next want to show that its image $W_{S/R}(\theta, \underline{g})$ is indepedent of the choice of the presentation θ and the choice of special sequence $\underline{g}$.

We first prove:

Lemma. *Let B be a ring, $J \subseteq B$ an ideal, and x, y elements of J that are nonzerodivisors in B. Then*

$$\frac{xB :_B J}{xB} \cong \frac{yB :_B J}{yB}$$

via the map that sends the class of $u \in xB :_B J$ to the class of an element $v \in yB :_B J$ such that $xv = yu$.

Proof. Given $u \in xB :_B J$, we have, since $y \in J$, that $yu \in xB$, and so $yu = xv$ with $v \in B$ (the choice of v is unique, since x is a nonzerodivisor in B). We first want to see that $v \in yB :_B J$, which means that if $a \in J$, then $av \in yB$. Since $a \in J$, $au = bx$ for $b \in B$. Then $auy = ybx$ and so $axv = ybx$. Since x is not a zerodivisor, this yields $av = yb$, as required. Next note that if we change the representative of the class of u, say to $u + xc$, then

$$y(u + xc) = yu + yxc = xv + yxc = x(v + yc).$$

Since v changes by a multiple of y, our map is well-defined. This establishes that we have a map

$$\frac{xB :_B J}{xB} \to \frac{yB :_B J}{yB}$$

of the form stated. By symmetry, there is a map

$$\frac{yB :_B J}{yB} \to \frac{xB :_B J}{xB}$$

of the same sort. By the symmetry of the condition $yu = xv$, if the class of u maps to the class of v then the class of v maps to the class of u, and vice versa. This shows that the two maps are mutually inverse. □

Theorem. *The image of the map $\Phi_{\theta,\underline{g}}$ in $\mathcal{L}$ is independent of the choice of $\underline{g}$, and of the choice of θ.*

Proof. To prove for a fixed presentation that the map is independent of the choice of special sequence suppose that we have two special sequences that yield maps with different images. We can preserve the fact that the images W, W' are different while localizing at a suitable prime or even maximal ideal of T: S is replaced by its localization at a corresponding prime. Simply choose the prime to be in the support of $(W+W')/(W\cap W')$. Thus, there is no loss of generality in assuming that T and S are local. The sequences in question remain special as we localize. But then, by the Lemma on comparison of special sequences from the beginning of the Lecture of October 9, we know that there exists a finite chain of special sequences joining the two that we are comparing such that any two consecutive sequences differ in at most one spot. Thus, we need only make the comparison when the two sequences differ in just one term, and since the sequences are permutable we may assume without loss of generality that one of them is $g_1 = g,\, g_2,\, \ldots,\, g_n$ and the other is $h,\, g_2,\, \ldots,\, g_n$, which we shall also denote $h_1, \ldots, h_n$.

We set up an isomorphism

$$\sigma\colon \frac{(g_1, \ldots, g_n)T :_T I}{(g_1, \ldots, g_n)T} \cong \frac{(h_1, \ldots, h_n)T :_T I}{(h_1, \ldots, h_n)T}$$

as follows. Let $B = R/(g_2, \ldots, g_n)$, let $J \subseteq B$ be the image of I, i.e., $I/(g_2, \ldots, g_n)$. Let x be the image of g and y the image of h. We now have the isomorphism by applying the Lemma proved just above.

To complete the proof of the independence of the image from the choice of special sequence we note that the following diagram commutes:

$$\begin{array}{ccc}
\dfrac{(g_1, \ldots, g_n)T :_T I}{(g_1, \ldots, g_n)T} & \xrightarrow{\ \sigma\ } & \dfrac{(h_1, \ldots, h_n)T :_T I}{(h_1, \ldots, h_n)T} \\
{\scriptstyle \Phi_{\theta,\underline{g}}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{\theta,\underline{h}}} \\
\mathcal{L} & \xrightarrow[\mathbf{1}_{\mathcal{L}}]{} & \mathcal{L}
\end{array}$$

To see this, one simply needs to see that if

$$(*) \qquad uh - vg = \sum_{j=2}^{n} t_j g_j$$

in T, then

$$(**) \qquad \frac{\overline{u}}{\gamma} = \frac{\overline{v}}{\eta}$$

in $\mathcal{L}$, where $\overline{u}$, $\overline{v}$ are the respective images of u and v in $\mathcal{L}$ and γ, η are the respective images of the determinants of the two Jacobian matrices in $\mathcal{L}$, i.e., that

$$u \det(\partial h_j/\partial X_i) \equiv v \det(\partial g_j/\partial X_i) \text{ modulo } I.$$

By differentiating $(*)$ with respect to each X_j in turn and using the fact that all the g, h and the g_j are in I, we see that, because the terms not shown coming from the product rule have a coefficient in I,

$$u\nabla h - v\nabla g \equiv \sum_{j=2}^{n} t_j \nabla g_j \quad \text{modulo } I.$$

Thus, the matrix whose columns are

$$u\nabla h,\, \nabla g_2,\, \dots\, ,\, \nabla g_n$$

and the matrix whose columns are

$$v\nabla g + \sum_{j=2}^{n} t_j \nabla g_j,\, \nabla g_2,\, \dots\, ,\, \nabla g_n$$

are equal mod I. By elementary column operations, we may drop the summation term from the first column of the second matrix when we calculate the determinant. Then we may factor u from the first column of the first matrix and v from the first column of the second matrix when we take determinants. This yields $\overline{u}\eta = \overline{v}\gamma$ in $\mathcal{L}$, and $(**)$ follows.

It remains only to prove that the image of $\Phi_{\theta,\underline{g}}$ is independent of the choice of $\theta\colon T \to S$ as well. We first consider the case of a finitely generated R-algebra S. The choice of a presentation is equivalent to the choice of a finite set of generators for S over R. We can compare the results from each of two different sets of generators with the result from their union, and so it suffices to see what happens when we enlarge a set of generators. By induction, it suffices to show that the image does not change when we enlarge a set of generators by one element, and so we may assume that we have $\theta : T = R[X_1, \dots , X_n] \twoheadrightarrow S$ and an extension of θ, $\theta'\colon T[X_{n+1}] \twoheadrightarrow S$ by sending X_{n+1} to s. Let $T' = T[X_{n+1}]$. We can choose an element $F \in T$ such that F maps to s in S, and it follows easily that the kernel I' of θ' is $I + (X_{n+1} - F)$. It also follows easily that if $g = g_1, \dots , g_n$ is special in I then $g' = g_1, \dots , g_{n+1}$ with $g_{n+1} = X_{n+1} - F$ is a special sequence in I'. The larger (size $n+1$) Jacobian matrix has the same determinant γ as the size n Jacobian matrix of $g_1, \dots , g_n$ with respect to $X_1, \dots , X_n$, and it is easy to check that there is an isomorphism

$$\tau\colon \frac{(g_1, \dots , g_n)T :_T I}{(g_1, \dots , g_n)T} \cong \frac{(g_1, \dots , g_{n+1})T' :_{T'} I'}{(g_1, \dots , g_{n+1})T'}$$

which is induced by the inclusion $(g_1, \dots , g_n)T :_T I \subseteq (g_1, \dots , g_{n+1})T' :_{T'} I'$. Since the Jacobian determinants are the same we have a commutative diagram

$$\begin{array}{ccc}
\dfrac{(g_1, \dots , g_n)T :_T I}{(g_1, \dots , g_n)T} & \xrightarrow{\ \tau\ } & \dfrac{(g_1, \dots , g_{n+1})T' :_{T'} I'}{(g_1, \dots , g_{n+1})T'} \\
{\scriptstyle \Phi_{\theta,g}}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi_{\theta',g'}} \\
\mathcal{L} & \xrightarrow[\mathbf{1}_{\mathcal{L}}]{} & \mathcal{L}
\end{array}$$

and this yields that the images are the same.

We have now justified the notation $W_{S/R}$ when S is finitely generated over R. We leave it to the reader as an exercise to verify that if s is a nonzerodivisor in S, then $W_{S[s^{-1}]/R} = (W_{S/R})_s$. Once we know this, by exactly the same argument we used to verify that the Jacobian ideal is independent of the choice of presentation for algebras essentially of finite type over R, it follows that $W_{S/R}(\theta)$ is independent of θ when S is essentially of finite type over R. $\square$

For a given special sequence $\underline{g}$ it is obvious from the definition of $\Phi_{\theta,\underline{g}}$ that γ multiplies the image of $\Phi_{\theta,\underline{g}}$ into $S \subseteq \mathcal{L}$. Since the image is independent of the choice of special sequence and since, by the Theorem on the existence of sufficiently many special sequences at the end of the Lecture Notes of October 6, as the special sequence varies the values of γ generate $J_{S/R}$, we have:

Corollary. $W_{S/R} \subseteq S :_{\mathcal{L}} \mathcal{J}_{S/R}$. $\square$

The following result gives several properties of $W_{S/R}$ that we will want to exploit.

Proposition. *Let S be generically étale, torsion-free and essentially of finite type over the Noetherian domain R. Let $W = W_{S/R}$.*

(a) *For any multiplicative system U in S, $W_{U^{-1}S/R} = U^{-1}W$.*

(b) *W is torsion-free over S.*

(c) *For every prime ideal P of S, if u, v is part of a system of parameters for S_P then it is a regular sequence on W_P. (Thus, W is S_2.)*

(d) *If $W \subseteq W' \subseteq \mathcal{L}$ and $W_P = W'_P$ for all height one primes of S and for all minimal primes of S that are also maximal ideals, then $W = W'$.*

(e) *If $R \to S$ is a local homomorphism of regular local rings then $\mathcal{J}_{S/R}$ is principal and $W = S :_{\mathcal{L}} \mathcal{J}_{S/R}$.*

(f) *If S is normal and R_P is regular for every prime ideal P of R lying under a height one prime ideal Q of S, then $W = S :_{\mathcal{L}} \mathcal{J}_{S/R}$.*

Proof. Part (a) is essentially the last part of (4.3), while (b) is evident from the fact that $W \subseteq \mathcal{L}$, by definition.

To prove (c) note that by (a) we may assume that S is local and that u, v is part of a system of parameters. We may choose a presentation $\theta : T \twoheadrightarrow S$ and think of W as $\cong \big((g_1, \ldots, g_n)T :_T I\big)/(g_1, \ldots, g_n)T$, where the sequence $g_1, \ldots, g_n$ is special. Let u_0, $v_0 \in T$ be representatives of u, v. Then $u_0 + I$ cannot be contained in the union of the minimal primes of $(g_1, \ldots, g_n)$, or else it will be contained in one of them by the Lemma on prime avoidance for cosets. Since this will contain I, it will be a minimal prime of I, and contradicts the statement that u is part of a system of parameters in $S = T/I$. Thus, we can replace u_0 by an element u_1 representing u such that $g_1, \ldots, g_n$, u_1 is part of a system of parameters for T. Similarly, $v_0 + I$ cannot be contained in the union of the minimal primes of $(g_1, \ldots, g_n, u_1)T$, or else it is contained in one of them, say Q. Thinking modulo I, we see that Q is a minimal prime of u in T/I containing v, a contradiction. Thus, we may choose u_1, v_1 in T representing u, v respectively and such that $g_1, \ldots, g_n$, u_1, v_1 is a regular sequence. Clearly, u_1, v_1 form a regular sequence on $T/(g_1, \ldots, g_n)T$. We claim they also form a regular sequence on the set of elements killed by I. It is clear that u_1 remains not a zerodivisor on this set. Suppose that $v_1 z = u_1 y$ where z, y are killed by I. Then $z = u_1 x$, $y = -v_1 x$ where, *a priori*, $x \in T/(g_1, \ldots, g_n)T$. But $Iz = 0$ and so $Iu_1 x = 0$, and since u_1 is not a zerodivisor on $T/(g_1, \ldots, g_n)T$, it follows that $Ix = 0$ as well.

Part (d) is a consequence of the result we proved in (c). If $W \neq W'$ we can localize at a minimal prime of the support of W'/W and preserve the counterexample. By hypothesis, this prime cannot have height one (nor height 0, since, if a height 0 prime is not maximal then we can localize at it in two steps: first localize at a height one prime that contains it). Thus, we may assume that S is local of height two or more, and that W'/W is a nonzero module of finite length. It follows that we can choose an element $x \in W' - W$ and part of a system of parameters u, v for S such that uz and vz are in W. The relations $v(uz) = u(vz)$ over W together with part (c) show that $uz \in uW$, and it follows that $z \in W$ after all, a contradiction.

To prove (e) note that when R is regular so is T, and so $T \to S$ will be a surjection of local rings. The kernel of such a surjection must be generated by part of a minimal set of generators for the maximal ideal of T. It follows that I is a prime and we have $I = (g_1, \ldots, g_n)T$ is itself generated by a suitable special sequence. Then $\mathcal{J}_{S/R}$ is generated by

$$\gamma = \det(\partial g_j / \partial X_i),$$

and

$$\frac{(g_1, \ldots, g_n)T :_T I}{(g_1, \ldots, g_n)T} = \frac{I :_T I}{I} = \frac{T}{I} = S$$

and Φ sends 1 to $\dfrac{1}{\gamma}$, so that $W = S\dfrac{1}{\gamma}$, and one sees that $S :_{\mathcal{L}} J_{S/R} = S :_{\mathcal{L}} \gamma S = W$, as claimed.

To prove (f) it suffices by (d) to consider the problem after localizing at a height one or zero prime Q of S, and, without affecting the issue, one may also localize R at its contraction. If the prime of S has height 0, so does its contraction to R, and both rings become regular after localization. If the prime of S has height one, then, again, both rings become regular after localization, S because it is normal and R by hypothesis. In either case the result follows from part (e). $\square$