Math 711: Lecture of October 13, 2006

A critical Lemma and the final step in the proof.

The following result of Lipman and Sathaye is exactly what is needed to establish that $W_{S/R}$ decreases as S is increased by adjoining integral fractions.

Lemma. *Let T be a commutative ring, Y an indeterminate, and J an ideal of $T[Y]$ such that J contains a monic polynomial h in Y of degree d. Suppose also that J contains an element of the form $\alpha Y - \beta$ where $\alpha,\, \beta \in T$ are such that $J :_{T[Y]} \alpha T[Y] = J$, i.e., such that α is not a zerodivisor modulo J. Let $\mathfrak{G} \subseteq T$ be an ideal of T with $\mathfrak{G} \subseteq J$. Then for every element $v \in T[Y]$ such that $vJ \subseteq (h,\, \mathfrak{G})T[Y]$ there exists an element $u \in T$ such that $u(J \cap T) \subseteq \mathfrak{G}$ and such that*

$$v \equiv u\,\frac{\partial h}{\partial Y} \text{ modulo } J.$$

Proof. We may replace T by $T/\mathfrak{G}$ and J by $J/\mathfrak{G}T[Y]$ without affecting the problem. Thus, we may assume without loss of generality that $\mathfrak{G} = (0)$. By the division algorithm we may replace v by its remainder upon division by h, and so assume that $v = \sum_{i=1}^{d} u_i Y^{d-i}$, where the $u_i \in T$. We shall show that we may take $u = u_1$, and we drop the subscript from here on. First note that $u_i(J \cap T) = (0)$ for all i: in fact since $vJ \subseteq hT[Y]$ (recall that we killed $\mathfrak{G}$) we have that $v(J \cap T)$ consists of multiples of h that have degree smaller than d, and these must be zero. In particular, $u(J \cap T) = (0)$. Likewise, $v(\alpha Y - \beta) \in hT[Y]$. The left hand side has degree at most d and coefficient αu in degree d, and so we must have $v(\alpha Y - \beta) = \alpha u h$. Differentiating with respect to Y yields

$$\frac{\partial v}{\partial Y}(\alpha Y - \beta) + v\alpha = \alpha u\,\frac{\partial h}{\partial Y}$$

and since $\alpha Y - \beta \in J$, we have that

$$\alpha(v - u\,\frac{\partial h}{\partial Y}) \equiv 0 \text{ modulo } J.$$

Since α is not a zerodivisor modulo J, the required result follows. $\square$

We now use this to prove:

Theorem. *If S_1 is obtained from S by adjoining finitely many integral fractions of $\mathcal{L}$, then $W_{S_1/R} \subseteq W_{S/R}$.*

Proof. By induction on the number of fractions adjoined, it is obviously sufficient to prove this when $S_1 = S[\lambda]$, where λ is a single element of $\mathcal{L}$. Choose a presentation $\theta\colon T \to S$ and a special sequence $g_1, \ldots, g_n$ in the kernel I. Let Y be a new indeterminate and extend θ to a map $T[Y] \twoheadrightarrow S[\lambda]$ by sending Y to λ. Since λ is integral over S there is a monic polynomial $h = h(Y) \in T[Y]$ of degree say, d, in the kernel J of $T[Y] \twoheadrightarrow S[\lambda]$. If $\lambda \in S$ there is nothing to prove so that we may assume that $d \geq 2$. Since λ is in $\mathcal{L}$ we may also choose α and β in T with α not a zerodivisor on I such that $\alpha Y - \beta$ is in the kernel. Consider the image of $h(Y)$ in $S[Y]$. There will be a certain subset of the minimal primes of S such that the image of λ is a multiple root of the image of h modulo those primes. If that set of primes is empty, we shall not alter h. If it is not empty choose an element of S that is not in any of those minimal primes but that is in the others, and represent it by an element $t \in T$. Then $h(Y) + t(\alpha Y - \beta)$ has the property that its image modulo any minimal prime of S has the image of λ as a simple root, and so we may assume, using this polynomial in place of the original choice of h, that h is a monic polynomial of degree $d \geq 2$ such that image of λ modulo every minimal prime of $\mathcal{L}$ is a simple root of the image of h.

Because h is monic in Y, the sequence $g_1, \ldots, g_n, h$ is a regular sequence, and the Jacobian determinant with respect to $X_1, \ldots, X_n, Y$ is $\gamma \dfrac{\partial h}{\partial Y}$, where γ is $\det(\partial g_j/\partial X_i)$. Our choice of h implies that $\dfrac{\partial h}{\partial Y}$ has image that is not in any minimal prime of $\mathcal{L}$, and it follows that $g_1, \ldots, g_n, h$ is a special sequence in J and can be used to calculate $W_{S[\lambda]/R}$. Let $v \in (g_1, \ldots, g_n, h)T[Y] :_{T[Y]} J$. We may now apply the preceding Lemma with this T, Y, J, v, α, β and h, while taking $\mathfrak{G} = (g_1, \ldots, g_n)T$. Note that $J \cap T = I$. Observe as well that v gives rise to a typical element of the module $W_{S[\lambda]/R}$, namely the image of $v/(\gamma \dfrac{\partial h}{\partial Y})$ in $\mathcal{L}$, We want to show that this element is in $W_{S/R}$. Pick u as in the preceding Lemma. Then $u \in (g_1, \ldots, g_n)T :_T I$ and since $v \equiv u \dfrac{\partial h}{\partial y}$ modulo J, this image is the same as the image of $(u \dfrac{\partial h}{\partial y})/(\gamma \dfrac{\partial h}{\partial y}) = u/\gamma$, and so is in $W_{S/R}$, as required. $\square$

The proof of the Jacobian theorem. We can now complete the proof of the theorem as already indicated in the Sketch of the proof of the Jacobian theorem in the Lecture of October 4. We know that S' is module-finite over S and can therefore be obtained from S by adjoining finitely many elements of the total quotient ring that are integral over S. We then have

$$S' :_{\mathcal{L}} \mathcal{J}_{S'/R} = W_{S'/R} \subseteq W_{S/R} \subseteq S :_{\mathcal{L}} \mathcal{J}_{S/R}$$

where the equality on the left follows from part (f) of the Proposition on p. 4 of the Lecture Notes of October 11, the middle inclusion follows from the Theorem above, and the inclusion on the right follows from the Corollary on p. 4 of the Lecture Notes of October 11. $\square$