Math 711: Lecture of October 23, 2006

One of our goals is to discuss what is known about the following conjecture of C. Lech, which has been an open question for over forty years.

Conjecture. *If $R \to S$ is a flat local map of local rings, then $e(R) \leq e(S)$.*

This is open even in dimension three when S is module-finite and free over R. Note that one can immediately reduce to the case where both rings are complete.

Under mild conditions, a local ring R of multiplicity 1 is regular: it suffices if the completion $\widehat{R}$ has no associated prime P such that $\dim(\widehat{R}/P) < \dim(\widehat{R})$. Therefore, the following result is related to Lech's conjecture:

Theorem. *If S is faithfully flat over R and S is regular then R is regular. In particular, if $(R, m, K) \to (S, \mathfrak{n}, L)$ is a flat local map of local rings and S is regular, then R is regular.*

Proof. The second statement implies the first, for if P is any prime of R then some prime Q of S lies over P, and we can apply the second statement to $R_P \to S_Q$ to conclude that R_P is regular.

To prove the second statement, let

$$(*) \quad \cdots \to G_n \to \cdots \to G_1 \to G_0 \to R \to R/m \to 0$$

be a minimal resolution of R/m over R. Then the matrix α_i of the map $G_i \to G_{i-1}$ has entries in m for all $i \geq 1$. Since S is R-flat, the complex obtained by applying $S \otimes_R \_$, namely

$$(**) \quad \cdots \to S \otimes_R G_n \to \cdots \to S \otimes_R G_1 \to S \otimes_R G_0 \to S/mS \to 0$$

gives an S-free resolution of S/mS over R. Moreover, the entries of the matrix of the map $S \otimes G_i \to S \otimes_R G_{i-1}$ are simply the images of the entries of the matrix α_i in S: these are in $\mathfrak{n}$, and so the complex given in $(**)$ is a *minimal* free resolution of S/mS over S. Thus, all of its terms are eventually 0, and this implies that all of the terms of $(*)$ are eventually 0. Hence, K has finite projective dimension over R, which implies that R is regular. $\square$

Before treating Lech's conjecture itself, we want to give several other characterizations of $e_I(M)$ when I is generated by a system of parameters. There is a particularly simple characterization in the Cohen-Macaulay case. We first recall some facts about regular sequences. The results we state in the Proposition below are true for an arbitrary regular sequence on an arbitrary module. However, we only indicate proofs for the situation where R is local, M is a finitely generated R-module, and $x_1, \ldots, x_d$ are elements of the maximal ideal of R. The proofs are valid whenever we are in a situation where regular sequences are permutable, which makes the arguments much easier. (There is a treatment of the case where the regular sequence is not assumed to be permutable in the Extra Credit problems in Problem Sets #2 and #3 from Math 615, Winter, 2004. It is assumed that $M = R$ there, but the proofs are completely unchanged in the module case.) Recall that, in all cases, by virtue of the definition, the fact that $x_1, \dots, x_d$ is a regular sequence on M implies that $(x_1, \dots, x_d)M \neq M$.

Proposition. *Let $x_1, \dots, x_d \in R$, let $I = (x_1, \dots, x_d)R$, and let M be an R-module.*

(a) *Let $t_1, \dots, t_d$ be nonnegative integers. Then $x_1, \dots, x_d$ is a regular sequence if and only if $x_1^{t_1}, \dots, x_d^{t_d}$ is a regular sequence on M.*

(b) *If $x_1, \dots, x_d$ is a regular sequence on M, and $a_1, \dots, a_d$ are nonnegative integers, then $x_1^{a_1} \cdots x_d^{a_d} w \in (x_1^{a_1+1}, \dots, x_d^{a_d+1})M$ implies that $w \in (x_1, \dots, x_d)M$.*

(c) *If $x_1, \dots, x_d$ is a regular sequence on M, $\mu_1, \dots, \mu_N$ are the monomials of degree n in $x_1, \dots, x_d$, and $w_1, \dots, w_N$ are elements of M such that $\sum_{j=1}^N \mu_j w_j \in I^{n+1}M$, then every $w_j \in IM$.*

(d) *If $x_1, \dots, x_d$ is a regular sequence on M, then $\mathrm{gr}_I(M)$ may be identified with*

$$(M/IM) \otimes_{R/I} (R/I)[X_1, \dots, X_d],$$

where the X_j are indeterminates and for nonnegative integers $a_1, \dots, a_d$ such that $\sum_{j=1}^d a_j = n$, the image of $x_1^{a_1} \cdots x_d^{a_d} M$ in $I^n M/I^{n+1}M$ corresponds to

$$(M/IM)X_1^{a_1} \cdots X_d^{a_d}.$$

(e) *If $x_1, \dots, x_d$ is a regular sequence on M, then $M/I^{n+1}M$ has a filtration in which the factors are $\binom{n+d}{d}$ copies of M/IM.*

Proof. (a) It suffices to prove the statement in the case where just one of t_i is different from 1: we can adjust the exponents on one element at a time. Since R-sequences are permutable, it suffices to do the case where only t_d is different from 1, and for this purpose we may work with $M/(x_1, \dots, x_{d-1})M$. Thus, we may assume that $d = 1$, and the assertion we need is that x^t is a nonzerodivisor if and only if x is. Clearly, if $xw = 0$ then $x^t w = 0$, while if $x^t w = 0$ for t chosen as small as possible and $w \neq 0$ then $x(x^{t-1}w) = 0$.

(b) If all the a_i are zero then we are already done. If not, we use induction on the number of $a_i > 0$. Since we are assuming a situation in which R-sequences on a module are permutable we may assume that $a_d > 0$. Then

$$x_1^{a_1} \cdots x_d^{a_d} w = \sum_{j=1}^{d-1} x_j^{a_j+1} w_j + x_d^{a_d+1} w_d$$

for elements $w_1, \ldots, w_d \in M$. Then

$$x_d^{a_d}(x_1^{a_1}\cdots x_{d-1}^{a_{d-1}}w - x_d w_d) \in (x_1^{a_1+1}, \ldots, x_{d-1}^{a_{d-1}+1})M,$$

and since $x_1^{a_1+1}, \ldots, x_{d-1}^{a_{d-1}}, x_d^{a_d}$ is also a regular sequence on M, we have that

$$x_1^{a_1}\cdots x_{d-1}^{a_{d-1}}w - x_d w_d \in (x_1^{a_1+1}, \ldots, x_{d-1}^{a_{d-1}+1})M.$$

This yields that

$$x_1^{a_1}\cdots x_{d-1}^{a_{d-1}}w \in (x_1^{a_1+1}, \ldots, x_{d-1}^{a_{d-1}+1}, x_d)M,$$

providing an example in which the number of $a_j > 0$ has decreased. This is a contradiction.

(c) Fix one of the $\mu_j = x_1^{a_1}\cdots x_d^{a_d}$. Then in every other μ_k and in every monomial of degree $n+1$, at least one x_i occurs with exponent a_i+1. Thus, $\mu_j w \in (x_1^{a_1+1}, \ldots, x_d^{a_d+1})M$, and $w_j \in IM$ by part (b).

(d) For each monomial $\widetilde{\mu}$ in $X_1, \ldots, X_d$ we write μ for the corresponding monomial in $x_1, \ldots, x_d$. We define a map from

$$I^nM \to \bigoplus_{\deg(\widetilde{\mu})=n} (M/IM)\widetilde{\mu}$$

by sending $\sum_i \mu_j w_j \mapsto \sum_j \widetilde{\mu_j}\overline{w}_j$, where $\overline{w_j}$ is the image of $w_j \in M$ in M/IM. This map is well-defined by part (c), and is obviously surjective. The elements of $I^{n+1}M$ are precisely those elements of M which can be represented as $\sum_i \mu_j w_j$ with every $w_j \in IM$, and it follows at once that the kernel of the map is $I^{n+1}M$.

(e) This follows at once from part (d), since we can initially use a filtration with factors $I^kM/I^{k+1}M$, $0 \le k \le n$, and then refine it because each of these splits into a direct sum of copies of M/IM such that the number of copies is the same as the number of monomials of degree k in $X_1, \ldots, X_d$. The number of monomials of degree at most d is $\binom{n+d}{d}$. $\square$

We next note:

Theorem. *If (R, m, K) is local of dimension d, M is Cohen-Macaulay of dimension d over R, $x_1, \ldots, x_d$ is a regular sequence on M, and $I = (x_1, \ldots, x_d)$, then $e_I(M) = \ell\big(M/(x_1, \ldots, x_d)M\big)$.*

Proof. By part (e) of the Lemma just above, we have that $M/I^{n+1}M$ has a filtration such that

(1) Every factor is $\cong M/IM$.

(2) The number of factors is $\binom{n+d}{d}$, i.e., is the same as the number of monomials of degree at most n in d indeterminates $X_1, \ldots, X_d$.

This gives the result, for we then have

$$\ell(M/I^{n+1}M) = \binom{n+d}{d}\ell(M/IM),$$

and the leading term of $\binom{n+d}{d}$ is $\frac{n^d}{d!}$. $\square$

Theorem. *Let $(R,\, m,\, K)$ be module-finite over a regular local ring A such that $x_1,\, \dots\, , x_d$ is a regular system of parameters of A, and let M be an R-module of dimension d. Let $I = (x_1,\, \dots\, , x_d)R$. Then $e_I(M)$ is the torsion-free rank of M over A.*

Proof. From the definition, it does not matter whether we think of M as an R-module, or whether we think of it as an A-module with maximal ideal $\mathfrak{n} = (x_1,\, \dots\, , x_d)A$. In the latter case, if ρ is the torsion-free rank of M as an A-module, we have an exact sequence of A-modules

$$0 \to A^\rho \to M \to C \to 0$$

where C is a torsion A-module, so that $\dim(C) < d$. It follows that

$$e_I(M) = e_{\mathfrak{n}}(M) = re_{\mathfrak{n}}(A) + 0 = r\ell\bigl(A/(x_1,\, \dots\, , x_d)A\bigr) = r \cdot 1 = r.$$

$\square$

Discussion. If R is equicharacteristic, we can always reach the situation of the Theorem above. The mulltiplicity does not change if we replace R by $\widehat{R}$. But then we can choose a coefficient field K, and the structure theorems for complete local rings guarantee that R is module-finite over $A = K[[x_1,\, \dots\, , x_d]] \subseteq \widehat{R}$.

More generally:

Theorem. *Let R be module-finite over a Cohen-Macaulay local ring B such that $x_1,\, \dots\, , x_d$ is a system of parameters for B. Let M be an R-module of dimension d. Let $I = (x_1,\, \dots\, , x_d)R$. If B is a domain, $e_I(M) = \ell(B/IB)\rho$, where ρ is the torsion-free rank of M over B. When B is not a domain, if there is a short exact sequence*

$$0 \to B^\rho \to M \to C \to 0$$

with $\dim(C) < d$, *then* $e_I(M) = \ell(B/IB)\rho$.

Proof. $\ell(M/I^{n+1})M$ is independent of whether one thinks of $x_1,\, \dots\, , x_d$ as in B or in R. Thus, we can replace R by B. The result is then immediate from our results on additivity of multiplicity and the fact that when B is Cohen-Macaulay, $e_I(B) = \ell(B/I)$. $\square$

We want to give a different characterization of multiplicities due to C. Lech. If $\underline{n} = n_1,\, \dots\, , n_d$ is a d-tuple of nonnegative integers and f is a real-valued function of $\underline{n}$, we write $\lim\limits_{\underline{n}\to\infty} f(\underline{n}) = r$, where $r \in \mathbb{R}$, to mean that for all $\epsilon > 0$ there exists N such that for all $\underline{n} = n_1, \ldots, n_d$ satisfying $n_i \geq N$, $1 \leq i \leq d$, we have that $|f(\underline{n}) - r| < \epsilon$. One might also write $\lim\limits_{\min \underline{n}\to\infty} f(\underline{n}) = r$ with the same meaning. If $\underline{x} = x_1, \ldots, x_d$ is a system of parameters for R, we temporarily define the *Lech multiplicity* $e^{\mathrm{L}}_{\underline{x}}(M)$ to be

$$\lim_{\underline{n}\to\infty} \frac{\ell\big(M/(x_1^{n_1}, \ldots, x_d^{n_d})M\big)}{n_1 \cdots n_d}.$$

We shall show that the limit always exists, is 0 if $\dim(M) < d$, and, with $I = (x_1, \ldots, x_d)R$, is $e_I(M)$ when $\dim(M) = d$.

We first prove:

Lemma. *Let $\underline{x} = x_1, \ldots, x_d$, $d \geq 1$, be a system of parameters for a local ring (R, m, K) and let M', M, and M'' be finitely generated R-modules. Given $\underline{n} = n_1, \ldots, n_d$, let $I_{\underline{n}} = (x_1^{n_1}, \ldots, x_d^{n_d})R$, and let $\mathcal{L}_{\underline{n}}(M) = \ell(M/I_{\underline{n}}M)/n_1 \cdots n_d$.*

(a) *If*

$$0 \to M' \to M \to M'' \to 0$$

is exact, then for any m-primary ideal J,

$$\ell(M''/JM'') \leq \ell(M/JM) \leq \ell(M'/JM') + \ell(M''/JM''),$$

i.e.,

$$0 \leq \ell(M/JM) - \ell(M''/JM'') \leq \ell(M'/JM').$$

Hence, for all $\underline{n}$,

$$\mathcal{L}_{\underline{n}}(M'') \leq \mathcal{L}_{\underline{n}}(M) \leq \mathcal{L}_{\underline{n}}(M') + \mathcal{L}_{\underline{n}}(M''),$$

i.e.,

$$0 \leq \mathcal{L}_{\underline{n}}(M) - \mathcal{L}_{\underline{n}}(M'') \leq \mathcal{L}_{\underline{n}}(M').$$

Therefore, if the three limits exist,

$$e^{\mathrm{L}}_{\underline{x}}(M'') \leq e^{\mathrm{L}}_{\underline{x}}(M) \leq e^{\mathrm{L}}_{\underline{x}}(M') + e^{\mathrm{L}}_{\underline{x}}(M''),$$

If $e^{\mathrm{L}}_{\underline{x}}(M') = 0$ and $e^{\mathrm{L}}_{\underline{x}}(M'') = 0$, then $e^{\mathrm{L}}_{\underline{x}}(M) = 0$.

(b) *If M has a finite filtration with factors N_j we have that for any m-primary ideal J, $\ell(M/JM) \leq \sum_j \ell(N_j/JN_j)$. Hence, for all $\underline{n}$, $\mathcal{L}_{\underline{n}}(M) \leq \sum_j \mathcal{L}_{\underline{n}}(N_j)$, and $e^{\mathrm{L}}_{\underline{x}}(M) = 0$ whenever $e^{\mathrm{L}}_{\underline{x}}(N_j) = 0$ for all j.*

(c) *If $\dim(M) < d$ then $e^{\mathrm{L}}_{\underline{x}}(M) = 0$.*

(d) *If $0 \to M' \to M \to M'' \to 0$ is exact and $\dim(M') < d$, then $e^{\mathrm{L}}_{\underline{x}}(M)$ and $e^{\mathrm{L}}_{\underline{x}}(M'')$ exist or not alike, and if they exist they are equal.*

(e) *If each of M and M' embeds in the other so that the cokernel has dimension $< d$, then $e^{\mathrm{L}}_{\underline{x}}(M)$ and $e^{\mathrm{L}}_{\underline{x}}(M')$ exist or not alike, and they are equal.*

Proof. Part (a) follows because we have an exact sequence of finite length modules

$$0 \to M'/(JM \cap M') \to M/JM \to M''/JM \to 0 \to 0$$

and $JM \cap M' \supseteq JM'$, so that

$$\ell\big(M'/(JM \cap M')\big) \geq \ell(M'/JM').$$

The remaining statements in part (a) follow at once.

Part (b) follows from part (a) by a straightforward induction on the length of the filtration.

To prove (c) we may use induction on d. If $d = 1$ then $\dim(M) = 0$, so that $\ell(M/I_{\underline{n}}M) = \ell(M)$ is constant for all sufficiently large $\underline{n}$, while the denominator $n_1 \to \infty$. If $d > 1$ we first take a finite prime cyclic filtration of M. Thus, we may assume without loss of generality that M is a prime cyclic module. If $\dim(M) = 0$, we again have a constant numerator and a denominator that $\to \infty$, and so we may assume $\dim(M) > 0$. Since the x_i generate a primary ideal, some x_i does not kill $M = R/Q$, and so is a nonzerodivisor on M. By renumbering, we assume that $i = d$. Consider $\underline{n} = n_1, \ldots, n_d$ and let $\underline{n}^- = n_1, \ldots, n_{d-1}$, let $\underline{x}^- = x_1, \ldots, x_{d-1}$, and let $I_{\underline{n}^-} = (x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}})R$. Then

$$\frac{M}{I_{\underline{n}}M} = \frac{M}{(I_{\underline{n}^-} + x_d^{n_d})M} \cong \frac{M/x_d^{n_d}M}{I_{\underline{n}^-}(M/x_d^{n_d}M)}.$$

Note that $M/x_d^{n_d}M$ has a filtration with n_d factor modules N_j, $0 \leq j \leq n_d - 1$, where $N_j = x_d^jM/x_d^{j+1}M \cong M/x_dM$, and so with $\overline{M} = M/x_dM$, we have that $\ell(M/I_{\underline{n}}M) \leq n_d\ell(\overline{M}/I_{\underline{n}^-}\overline{M})$. It follows that

$$(*) \qquad \frac{\ell(M/I_{\underline{n}}M)}{n_1 \cdots n_d} \leq \frac{n_d\ell(\overline{M}/I_{\underline{n}^-}\overline{M})}{n_1 \cdots n_{d-1}n_d} = \frac{\ell(\overline{M}/I_{\underline{n}^-}\overline{M})}{n_1 \cdots n_{d-1}}.$$

We may view $\overline{M}$ as a module over $R/x_d^{n_d}R$, and

$$\dim(\overline{M}) < \dim(M) \leq \dim(R) - 1 = \dim(R/x_d^{n_d}).$$

By the induction hypothesis, $e^{\mathrm{L}}_{\underline{x}^-}(\overline{M}) = 0$ (working over $R/x_d^{n_d}R$), and it follows from $(*)$ that $e^{\mathrm{L}}_{\underline{x}}(M) = 0$ as well.

For part (d), note that (a) implies that $|\mathcal{L}_{\underline{n}}(M) - \mathcal{L}_{\underline{n}}(M'')| \leq \mathcal{L}_{\underline{n}}(M')$, and we are assuming that $\mathcal{L}_{\underline{n}}(M') \to 0$ as $\underline{n} \to \infty$.

Finally, for part (e), note that if we have short exact sequences

$$0 \to M' \to M \to C_1 \to 0 \quad \text{and} \quad 0 \to M \to M' \to C_2 \to 0$$

then from the first we have $\mathcal{L}_{\underline{n}}(M) - \mathcal{L}_{\underline{n}}(M') \leq \mathcal{L}_{\underline{n}}(C_1)$ and from the second we have $\mathcal{L}_{\underline{n}}(M') - \mathcal{L}_{\underline{n}}(M) \leq \mathcal{L}_{\underline{n}}(C_2)$. Hence, $|\mathcal{L}_{\underline{n}}(M) - \mathcal{L}_{\underline{n}}(M')| \leq \max\{\mathcal{L}_{\underline{n}}(C_1), \mathcal{L}_{\underline{n}}(C_2)\} \to 0$ as $\underline{n} \to \infty$. $\square$