Math 711: Lecture of October 25, 2006

Proposition. *Let M be an R-module. Let*

(a) *If M has a finite filtration with factors N_j, $1 \leq j \leq s$, and x is a nonzerodivisor on every N_j, then M/xM has a filtration with s factors N_j/xN_j, and M/x^nM has a filtration with ns factors: there are n copies of every N_j/xN_j, $1 \leq j \leq s$.*

(b) *If $x_1, \ldots, x_d$ is a regular sequence on M and $n_1, \ldots, n_d$ are nonnegative integers, then $M/(x_1^{n_1}, \ldots, x_d^{n_d})M$ has a filtration by $n_1 \cdots n_d$ copies of $M/(x_1, \ldots, x_d)M$.*

Proof. (a) By induction on the number of factors, this comes down to the case where there are two factors. That is, one has $0 \to N_1 \to M \to N_2 \to 0$. This has an isomorphic subcomplex $0 \to xN_1 \to xM \to xN_2 \to 0$, and the desired statement now follows from the exactness of the quotient complex. It follows as well that M/x^nM has a filtration by the modules N_j/x^nN_j, and each of these has a filtration with n factors, $x^kN_j/x^{k+1}N_j \cong N_j/xN_j$, $0 \leq k \leq n-1$.

For part (b) we use induction on d. The case $d = 1$ has already been handled in part (a). For the inductive step, we know that $M/(x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}})M$ has a filtration by $n_1 \cdots n_{d-1}$ copies of $M/(x_1, \ldots, x_{d-1})M$, and $x = x_d$ is a nonzerodivisor on each of these. The result now follows from the last statement in part (a), with $n = n_d$. $\square$

We next observe:

Lemma. *Let R be a Noetherian ring and let M be a finitely generated R-module of dimension $d > 0$.*

(a) *M contains a maximum submodule N such that $\dim(N) < d$, and M/N has pure dimension d, i.e., for every $P \in \mathrm{Ass}(M/N)$, $\dim(R/P) = d$.*

(b) *Let W be a multiplicative system of R consisting of nonzerodivisors and suppose that M and M' are R-modules such that $W^{-1}M \cong W^{-1}M'$. Then there exist exact sequences $0 \to M' \to M \to C_1 \to 0$ and $0 \to M \to M' \to C_2 \to 0$ such that each of C_1 and C_2 is killed by a single element of W.*

(c) *Let (R, m, K) be a complete local ring of dimension d, and let M be a finitely generated faithful R-module of pure dimension d. Let $x_1, \ldots, x_d$ be a system of parameters for R. If R contains a field there is a coefficient field $K \subseteq R$ for R, and M is a torsion-free module over $A = K[[x_1, \ldots, x_d]]$, so that for some integer $\rho > 0$, M and A^ρ become isomorphic when we localize at $W = A - \{\{0\}$.*

In mixed characteristic, there exists Cohen-Macaulay ring $A \subseteq R$ containing $x_1, \ldots, x_d$ as a system of parameters such that A has the form $B/(f)$ where B is regular and $f \neq 0$. Moreover, if W is the multiplicative system of nonzerodivisors in A then W consists of nonzero divisors of on M and $W^{-1}M$ is a finite direct sum of modules of *the form $W^{-1}B/g_jB$ where each g_j is a divisor of f. In particular, $M' = \bigoplus_j B/g_jB$ is a Cohen-Macaulay module over A of pure dimension d such that $W^{-1}M$ and $W^{-1}M'$ are isomorphic as A-modules.*

Proof. To prove (a), first note that since M has ACC on submodules, it has a maximal submodule N of dimension less than d: it may be 0. If N' is another submodule of M of dimension $< d$, then $d > \dim(N \oplus N') \geq \dim(N + N')$, and so $N + N' \subseteq M$ contradicts the maximality of N. Thus, N contains every submodule of M of dimension $< d$. If M/N had any nonzero submodule of dimension less than d, its inverse image in M would be strictly larger than N and of dimension less than d as well.

(b) Since $M \subseteq W^{-1}M \cong W^{-1}M'$, we have an injection $M \hookrightarrow W^{-1}M'$. Let $u_1, \ldots, u_h$ be generators of M. Suppose that u_i maps to v_i/w_i, $1 \leq i \leq h$, where $v_i \in M'$ and $w_i \in W$. Let $w = w_1 \cdots w_h$. Then $M \cong wM \hookrightarrow \sum_i Rv_i \subseteq M'$. The map $W^{-1}M \to W^{-}M'$ that this induces is still an isomorphsim, since w is a unit in $W^{-1}R$. It follows that the cokernel C_1 of the map $M \to M'$ that we constructed is such that $W^{-1}C_1 = 0$. Since C_1 is finitely generated, there is a single element of W that kills C_1. An entirely similar argument yields $0 \to M' \to M \to C_2$ such that C_2 is killed by an element of W.

(c) Let $u_1, \ldots, u_h$ generate M. Then the map $R \to M^{\oplus h}$ sending $r \mapsto (ru_1, \ldots, ru_h)$ is injective. It follows that $\mathrm{Ass}(R) \subseteq \mathrm{Ass}(M^{\oplus h}) = \mathrm{Ass}(M)$, so that R is also of pure dimension d. Choose a field or discrete valuation ring V that maps onto a coefficient ring for R (so that the residue class field of V maps isomorphically to the residue class field of R), and let $X_1, \ldots, X_d$ be formal indeterminates over V. Then the map $V \to R$ extends uniquely to a continuous map $B = V[[X_1, \ldots, X_d]] \to R$ such that $X_i \mapsto x_i$, $1 \leq i \leq d$. Let M_B be the maximal ideal of B. Since the map $B \to R$ induces an isomorphism of residue class fields, and since R/m_BR has finite length over B (the x_i generate an m-primary ideal of R), R is module-finite over the image A of B in R. Moreover, we must have $\dim(A) = \dim(R)$.

In the equal characteristic case, where $V = K$ is a field, we must have $B \cong A = K[[x_1, \ldots, x_d]]$. Moreover, M must be torision-free over A, since a nonzero torsion submodule would have dimension smaller than d. Hence M and A^ρ, where ρ is the torsion-free rank of M over A, become isomorphic when we localize at $A - \{0\}$.

We suppose henceforth that we are in the mixed characteristic case. We know that the ring A has pure dimension d. It follows that $A = B/J$, where J is an ideal all of whose associated primes in B have height one. Since B is regular, it is a UFD. Height one primes are principal, and any ideal primary to a height one prime has the form g^k, where g generates the prime and k is a nonnegative integer. It follows that $A = B/fB$, where $f = f_1^{k_1} \cdots f_h^{k_k}$ is the factorization of f into prime elements. Let W be the multiplicative system consisting of the complement of the union of the f_jB. The associated primes of A are the $P_j = f_jA$, and these are also the associated primes of M. Then $W^{-1}A$ is an Artin ring and is the product of the local rings A_{Pj}: each of these may be thought of as obtained by killing $f_j^{k_j}$ in the DVR obtained by localizing B at the prime f_jB. M, as a B-module, is then a product of modules over the various A_{P_j}, each of which is a direct sum of cyclic modules of the form $B/f_j^s B$ for $1 \le s \le k_j$. Each of these is Cohen-Macaulay of dimension d, and the images the x_i form a system of parameters, since each of these rings is a homomorphic image of A. $\square$

We can now prove:

Theorem. *Let (R, m, K) be a local ring of dimension d and let $x_1, \ldots, x_d$ be a system of parameters for R. Let $I = (x_1, \ldots, x_d)R$. Let M be a finitely generated R-module. Then $e^{\mathrm{L}}_{\underline{x}}(M) = 0$ if $\dim(M) < d$, and $e^{\mathrm{L}}_{\underline{x}}(M) = e_I(M)$ if $\dim(M) = d$.*

Proof. We have already proved that $e^{\mathrm{L}}_{\underline{x}}(M) = 0$ if $\dim(M) < d$: this is part (c) of the Lemma on p. 5 of the Lecture Notes from October 23. Now suppose that $\dim(M) = d$. We may complete R and M without changing either multiplicity. Let N be the maximum submodule of M of dimension smaller than d. Then we may replace M by M/N (cf. part (d) of the Lemma on p. 5 of the the Lecture Notes from October 23). Thus, we may assume that M has pure dimension d. We may replace R by $R/\mathrm{Ann}_R M$ and so assume that M is faithful. We view R as module-finite over A as in part (c) of the preceding Lemma. Since A contains $x_1, \ldots, x_d$, we may replace R by A and I by $(x_1, \ldots, x_d)A$. By parts (b) and (c) of the preceding Lemma, there is a Cohen-Macaulay A-module M' of dimension d such that each of M and M' embeds in the other with cokernel of dimension smaller than d. Thus, by part (e) of the Lemma on p. 5 of the Lecture Notes of October 23, we need only prove the result for M'. Hence, we may assume that M is Cohen-Macaulay. But it follows from the part (b) of the Proposition at the beginning of this lecture that $e^{\mathrm{L}}_{\underline{x}}(M) = \ell(M/IM)$ when M is Cohen-Macaulay, and we also know that $e_I(M) = \ell(M/M)$ in this case. $\square$

We next review the definition and some basic properties of the Koszul complex

$$\mathcal{K}_\bullet(x_1, \ldots, x_n;\, M),$$

where $x_1, \ldots, x_n \in R$ and M is an R-module.

We first consider the case where $M = R$. We let $\mathcal{K}_1(x_1, \ldots, x_n;\, R)$ be the free module G with free basis $u_1, \ldots, u_n$. As a module, we let $\mathcal{K}_i(x_1, \ldots, x_n;\, R)$ be the free module $\bigwedge^i(G)$, which has a free basis with $\binom{n}{i}$ generators $u_{j_1} \wedge \cdots \wedge u_{j_i}$, $j_1 < \cdots < j_i$. The differential is such that $du_i = x_i$. More generally, the formula for the differential d is

$$(*) \quad d(u_{j_1} \wedge \cdots \wedge u_{j_i}) = \sum_{t=1}^{i} (-1)^{t-1} x_{j_t} u_{j_1} \wedge \cdots \wedge u_{j_{t-1}} \wedge u_{j_{t+1}} \cdots \wedge u_{j_i}.$$

Consider an $\mathbb{N}$-graded skew-commutative R-algebra Λ. (This is an $\mathbb{N}$-graded associative algebra with identity such that for any two forms of degree f, g of degree h and k respectively, $gf = (-1)^{hk} gf$. That is, elements of even degree are in the center, and multiplying two elements of odd degree in reverse order reverses the sign on the product). An R-linear map d of Λ into itself that lowers degrees of homogeneous elements by one and satisfies

$$(\#) \quad d(uv) = (du)v + (-1)^{\deg(u)} u\, dv$$

when u is a form is called an R-*derivation* of Λ.

Then $\bigwedge^\bullet(G)$ is an N-graded skew-commutative R-algebra, and it is easy to verify that the differential is an R-derivation. By the R-bilinearity of both sides in u and v, it suffices to verify $(\#)$ when $u = u_{j_1} \wedge \cdots \wedge u_{j_h}$ and $v = u_{k_1} \wedge \cdots \wedge u_{k_i}$ with $j_1 < \cdots < j_h$ and $k_1 < \cdots < k_i$. It is easy to see that this reduces to the assertion $(**)$ that the formula $(*)$ above is correct even when the sequence $j_1, \ldots, j_i$ of integers in $\{1, 2, \ldots, n\}$ is allowed to contain repetitions and is not necessarily in ascending order: one then applies $(**)$ to $j_1, \ldots, j_h, k_1, \ldots, k_i$. To prove $(**)$, note that if we switch two consecutive terms in the sequence $j_1, \ldots, j_i$ every term on both sides of $(*)$ changes sign. If the $j_1, \ldots, j_i$ are mutually distinct this reduces the proof to the case where the elements are in the correct order, which we know from the definition of the differential. If the elements are not all distinct, we may reduce to the case where $j_t = j_{t+1}$ for some t. But then $u_{j_1} \wedge \cdots \wedge u_{j_i} = 0$, while all but two terms in the sum on the right contain $u_{j_t} \wedge u_{j_{t+1}} = 0$, and the remaining two terms have opposite sign.

Once we know that d is a derivation, we obtain by a straightforward induction on k that if $v_1, \ldots, v_k$ are forms of degrees $a_1, \ldots, a_k$, then

$$(***) \quad d(v_1 \wedge \cdots \wedge v_i) = \sum_{t=i} (-1)^{a_1 + \cdots + a_{t-1}} v_{j_1} \wedge \cdots \wedge v_{j_{t-1}} \wedge dv_{j_t} \wedge v_{j_{t+1}} \wedge \cdots \wedge v_{j_i}.$$

Note that the formula $(*)$ is a special case in which all the given forms have degree 1.

It follows that the differential on the Koszul complex is uniquely determined by what it does in degree 1, that is, by the map $G \to R$, where G is the free R-module $\mathcal{K}_1(\underline{x}; R)$, together with the fact that it is a derivation on $\bigwedge(G)$. Any map $G \to R$ extends uniquely to a derivation: we can choose a free basis $u_1, \ldots, u_n$ for G, take the x_i to be the values of the map on the u_i, and then the differential on $\mathcal{K}_\bullet(x_1, \ldots, x_n; R)$ gives the extension we want. Uniqueness follows because the derivation property forces $(***)$ to hold, and hence forces $(*)$ to hold, thereby determining the values of the derivation on an R-free basis.

Thus, instead of thinking of the Koszul complex $\mathcal{K}(x_1, \ldots, x_n; R)$ as arising from a sequence of elements $x_1, \ldots, x_n$ of R, we may think of it as arising from an R-linear map of a free module $\theta : G \to R$ (we might have written d_1 for θ), and we write $\mathcal{K}_\bullet(\theta; R)$ for the corresponding Koszul complex. The sequence of elements is hidden, but can be recovered by choosing a free basis for G, say $u_1, \ldots, u_n$, and taking $x_i = \theta(u_i)$, $1 \le i \le n$. The exterior algebra point of view makes it clear that the Koszul complex does not depend on the choice of the sequence of elements: only on the map of the free module $G \to R$. Different choices of basis produce Koszul complexes that look different from the "sequence of elements" point of view, but are obviously isomorphic. In particular, up to isomprphism, permuting the elements does not change the complex.

We write $\mathcal{K}_\bullet(x_1, \dots, x_d;\, M)$, where M is an R-module, for $\mathcal{K}_\bullet(x_1, \dots, x_n;\, R) \otimes M$. The homology of this complex is denoted $H_\bullet(x_1, \dots, x_n;\, M)$. Let $\underline{x} = x_1, \dots, x_n$ and Let $I = (\underline{x})R$.

We have the following comments:

(1) The complex is finite: if M is not zero, it has length n. The i the term is the direct sum of $\binom{n}{i}$ copies of M. Both the complex and its homology are killed by $\mathrm{Ann}_R M$.

(2) The map from degree 1 to degree 0 is the map $M^n \to M$ sending

$$(u_1, \dots, u_n) \mapsto x_1u_1 + \cdots x_nu_n.$$

The image of the map is IM, and so $H_0(x_1, \dots, x_d;\, M) \cong M/IM$.

(3) The map from degree n to degree $n-1$ is the map $M \to M^n$ that sends

$$u \mapsto (x_1u_1,\, -x_2u_2, \cdots, \pm x_nu_n),$$

and so $H_n(x_1, \dots, x_n;\, M) \cong \mathrm{Ann}_M I$.

(4) Given a short exact sequence of modules $0 \to M' \to M \to M'' \to 0$ we may tensor with the free complex $\mathcal{K}_\bullet(x_1, \dots, x_n;\, R)$ to obtain a short exact sequence of complexes

$$\mathcal{K}_\bullet(x_1, \dots, x_n;\, M') \to \mathcal{K}_\bullet(x_1, \dots, x_n;\, M) \to \mathcal{K}_\bullet(x_1, \dots, x_n;\, M'') \to 0.$$

The snake lemma then yields a long exact sequence of Koszul homology:

$$\cdots \to H_i(\underline{x};\, M') \to H_i(\underline{x};\, M) \to H_i(\underline{x};\, M'') \to H_{i-1}(\underline{x};\, M') \to \cdots$$

$$\to H_1(\underline{x};\, M') \to H_1(\underline{x};\, M) \to H_1(\underline{x};\, M'') \to M'/IM' \to M/IM \to M''/IM'' \to 0$$

(5) I kills $H_i(\underline{x};\, M)$ for every i and every R-module M. It suffices to see that x_d kills the homology: the argument for x_i is similar. Let $z \in \mathcal{K}_i(\underline{x};\, M)$, and consider $z \wedge u_n \in \mathcal{K}_{i+1}(\underline{x};\, M)$. Then

$$(*) \quad d(z \wedge u_n) = dz \wedge u_n + (-1)^i x_n z.$$

Hence, if z is a cycle, $d(z \wedge u_n) = (-1)^i x_n z$, which shows that $x_n z$ is a boundary.

(6) Let $\underline{x}^-$ denote $x_1, \dots, x_{n-1}$. Let $G^- \subseteq G$ be the free module on the free basis $u_1, \dots, u_{n-1}$. Then $\mathcal{K}_\bullet(\underline{x}^-;\, M)$ may be identified with

$$\textstyle\bigwedge^\bullet(G^-) \otimes_R M \subseteq \bigwedge^\bullet(G) \otimes M = \mathcal{K}_\bullet(\underline{x};\, M).$$

This subcomplex is spanned by all terms that involve only $u_1, \dots, u_{n-1}$. The quotient complex my be identified with $\mathcal{K}_\bullet(\underline{x}^-; M)$ as well: one lets $u_{j_1} \wedge \cdots \wedge u_{j_{i-1}} \wedge u_n \otimes w$

in degree i, where the $j_\nu < n$ and $w \in W$, correspond to $u_{j_1} \wedge \cdots \wedge u_{j_{i-1}} \otimes w$ in degree $i-1$. This gives a short exact sequence of complexes

$$0 \to \mathcal{K}_\bullet(\underline{x}^-;\, M) \to \mathcal{K}_\bullet(\underline{x};\, M) \to \mathcal{K}_{\bullet-1}(\underline{x}^-;\, M).$$

This in turn leads to a long exact sequence for homology:

$$\cdots \to H_i(\underline{x}^-;\, M) \to H_i(\underline{x}^-;\, M) \to H_i(\underline{x};\, M) \to H_{i-1}(\underline{x}^-;\, M) \to H_{i-1}(\underline{x}^-;\, M) \to \cdots.$$

The maps $\delta_i : H_i(\underline{x}^-;\, M) \to H_i(\underline{x}^-; M)$ and $\delta_{i-1} : H_{i-1}(\underline{x}^-;\, M) \to H_{i-1}(\underline{x}^-; M)$ are connecting homomorphisms. They may be computed as follows: a cycle z in the homology of the quotient complex $\mathcal{K}_\bullet(\underline{x}^-;\, M)$ in degree i can be lifted to $\mathcal{K}_{i+1}(\underline{x};\, M)$ as $z \wedge u_n$, and the differential takes this to $(-1)^i x_n z$ by the argument given in (5). Hence, δ_i is the endomorphism given by multiplication by $(-1)^i x_i$. It follows that we have short exact sequences:

$$0 \to \frac{H_i(\underline{x}^-;\, M)}{x_n H_i(\underline{x}^-;\, M)} \to H_i(\underline{x};\, M) \to \mathrm{Ann}_{H_{i-1}(\underline{x}^-;\, M)} x_n \to 0$$

for every i.

We next want to show that multiplicities with respect to a system of parameters can be computed using Koszul homology. Note that the matrices of the maps in the Koszul complex $\mathcal{K}_\bullet(\underline{x}; M)$ have entries in $I = (\underline{x})R$, so that for all every $\mathcal{K}_i(\underline{x};\, M)$ maps into $I\mathcal{K}_{i-1}(\underline{x};\, M)$ and for all s, $I^s\mathcal{K}_i(\underline{x};\, M)$ maps into $I^{s+1}\mathcal{K}_{i-1}(\underline{x};\, M)$.

Theorem. *Let M be a finitely generated module over a Noetherian ring R, let $\underline{x} = x_1, \ldots, x_n \in R$ and let $I = (\underline{x})R$. Then for all sufficiently large $h \gg n$, the subcomplex*

$$0 \to I^{h-n}\mathcal{K}_n(\underline{x};\, M) \to \cdots \to I^{h-i}\mathcal{K}_i(\underline{x};\, M) \to \cdots \to I^{h-1}\mathcal{K}_1(\underline{x};\, M) \to I^h\mathcal{K}_0(\underline{x};\, M) \to 0$$

of the Koszul complex $\mathcal{K}_\bullet(\underline{x};\, M)$ is exact (not just acyclic).

Proof. We abbreviate $\mathcal{K}_i = \mathcal{K}_i(\underline{x};\, M)$. Since there are only finitely many spots where the complex is nonzero, the assertion is equivalent to the statement that for fixed i, every cycle in $I^{k+1}\mathcal{K}_i$ is the boundary of an element in $I^k\mathcal{K}_{i+1}$ for all $k \gg 0$.

Let Z_i denote the module of cycles in $\mathcal{K}_i$. By the Artin-Rees lemma, there is a constant c_i such that for all $k \geq c_i$, $I^k\mathcal{K}_i \cap Z_i = I^{k-c_i}(I^{c_i}\mathcal{K}_i \cap Z_i)$. In particular, for all $k \geq c_i$, $I^{k+1}\mathcal{K}_i \cap Z_i = I(I^k\mathcal{K}_i \cap Z_i)$. For any k, the complex $I^k\mathcal{K}_\bullet(\underline{x};\, M)$., i.e.,

$$0 \to I^k\mathcal{K}_n(\underline{x};\, M) \to \cdots \to I^k\mathcal{K}_i(\underline{x};\, M) \to \cdots \to I^k\mathcal{K}_1(\underline{x};\, M) \to I^k\mathcal{K}_0(\underline{x};\, M) \to 0,$$

is the same as $\mathcal{K}_\bullet(\underline{x};\, I^kM)$, and so its homology is killed by I. Thus, a cycle in $I^k\mathcal{K}_i$, which is the same as an element of $I^k\mathcal{K}_i \cap Z_i$, when multiplied by any element of I, is a boundary. But for $k \gg 0$, $I^{k+1}\mathcal{K}_i \cap Z = I(I^k\mathcal{K} \cap Z)$, which is in the image of $I^k\mathcal{K}_{i+1}$, as required. $\square$