Math 711: Lecture of October 27, 2006

Discussion: the difference operator. Consider the ring $\mathbb{Q}[n]$ of polynomials in one variable n over the rational numbers. We define a $\mathbb{Q}$-linear function τ from this ring to itself by $\tau\big(P(n)\big) = P(n-1)$. Note that τ preserves degree and leading term. We write $\mathbf{1}$ for the identity map on $\mathbb{Q}[n]$, and Δ for the operator $\mathbf{1} - \tau$ that sends $P(n) \mapsto P(n) - P(n-1)$. Note that Δ lowers degree by one (if the degree is positive) and kills scalars. Moreover, if the leading term of $P(n)$ is an^d, where $a \in \mathbb{Q}$, the leading term of $\Delta\big(P(n)\big)$ is adn^{d-1}, which is similar to the behavior of the differentiation operator. In particular, if $P(n)$ has degree d, $\Delta^d P(n)$ is the scalar $d!a$, where a is the leading coefficient of $P(n)$. For each constant integer $c \geq 0$, $\tau^c\big(P(n)\big) = P(n-c)$. By the binomial theorem, for each k the operator

$$\Delta^k = (\mathbf{1} - \tau)^k = \mathbf{1} - \binom{k}{1}\tau + \binom{k}{2}\tau^2 - \cdots + (-1)^k\binom{k}{k}\tau^k$$

so that

$$(\#) \quad \Delta^k\big(P(n)\big) = P(n) - kP(n-1) + \cdots + (-1)^i\binom{k}{i}P(n-i) + \cdots + (-1)^k P(n-k).$$

We also note:

Lemma. *If $0 \to N_b \to \cdots \to N_a \to 0$ is a bounded complex of modules of finite length, the alternating sum of the lengths $\sum_{i=a}^b (-1)^i \ell(N_i)$ is the same as $\sum_{i=a}^b (-1)^i \ell\big(H_i(N_\bullet)\big)$.*

Proof. Let B_i be the image of N_{i+1} in N_i and Z_i the kernel of $N_i \to N_{i-1}$, so that $H_i = H_i(N_\bullet) = Z_i/B_i$. Then we have short exact sequences

$$0 \to Z_i \to N_i \to B_{i-1} \to 0 \quad \text{and} \quad 0 \to B_i \to Z_i \to H_i \to 0$$

for all i. It will be convenient to think of our summations as taken over all integers $i \in \mathbb{Z}$: this still makes sense since all but finitely many terms are zero, and will permit a convenient shift in the summation index. We then have:

$$\sum_i (-1)^i \ell(H_i) = \sum_i (-1)^i \big(\ell(Z_i) - \ell(B_i)\big) = \sum_i (-1)^i \ell(Z_i) + \sum_i (-1)^{i+1} \ell(B_i) =$$

$$\sum_i (-1)^i \ell(Z_i) + \sum_i (-1)^i \ell(B_{i-1}) = \sum_i (-1)^i \big(\ell(Z_i) + \ell(B_{i-1})\big) = \sum_i (-1)^i \ell(N_i).$$

$\square$

If (R, m, K) is local and $x_1, \ldots, x_d$ is a system of parameters, then for any finitely generated R-module M, all the modules $H_i(x_1, \ldots, x_d; M)$ have finite length: each is a finitely generated module killed by $(x_1, \ldots, x_d)R$.

Theorem (Serre). *Let (R, m, K) be a local ring of Krull dimension d, and let M be a finitely generated R-module. Let $I = (x_1, \ldots, x_d)R$. Then*

$$\sum_{i=0}^{d}(-1)^i \ell\big(H_i(x_1, \ldots, x_d; M)\big)$$

is $e_I(M)$ if $\dim(M) = d$ and is 0 if $\dim(M) < d$.

Proof. By the Theorem at the end of the Lecture Notes of October 25, the subcomplex $\mathcal{A}_\bullet^{(n)}$ whose ith term is $I^{n-i}\mathcal{K}_i$, where $\mathcal{K}_i = \mathcal{K}_i(x_1, \ldots, x_d; M)$, is exact for all $n \gg 0$. Call the quotient complex $\mathcal{Q}_\bullet^{(n)}$. The long exact sequence of homology coming from the short exact sequence

$$0 \to \mathcal{A}_\bullet^{(n)} \to \mathcal{K}_\bullet \to \mathcal{Q}_\bullet^{(n)} \to 0$$

shows that $H_i(x_1, \ldots, x_d; M) \cong H_i(\mathcal{Q}_\bullet^{(n)})$ for all i if $n \gg 0$. Let $H(n)$ denote the Hilbert polynomial of M with respect to I, which agrees with $\ell(M/I^{n+1}M)$ for all $n \gg 0$. Then

$$\sum_{i}(-1)^i \ell\big(H_i(x_1, \ldots, x_d; M)\big)$$

is the same as

$$\sum_{i}(-1)^i \ell\big(H_i(\mathcal{Q}_\bullet^{(n+1)})\big)$$

for all $n \gg 0$, and this in turn equals

$$\sum_{i}(-1)^i \ell(\mathcal{Q}_i^{(n+1)})$$

by the Lemma just above. Since $\mathcal{Q}_i^{(n+1)}$ is the direct sum of $\binom{d}{i}$ copies of $M/I^{n+1-i}M$, for $n \gg 0$ this is

$$\sum_{i}(-1)^i \binom{d}{i} H(n-i),$$

which is $\Delta^d\big(H(n)\big)$ by the formula (#) in the Discussion at the beginning of this Lecture. By that same Discussion, this is also $d!$ times the leading coefficient of $H(n)$. $\square$

Discussion: mapping cones. Let $\phi_\bullet : A_\bullet \to B_\bullet$ be a map of complexes of R-modules, so that we have a commutative diagram:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_{i+1}} & B_i & \xrightarrow{d_i} & B_{i-1} & \xrightarrow{d_{i-1}} & \cdots \\
& & \phi_i \big\uparrow & & \phi_{i-1} \big\uparrow & & \\
\cdots & \xrightarrow{\delta_{i+1}} & A_i & \xrightarrow{\delta_i} & A_{i-1} & \xrightarrow{\delta_{i-1}} & \cdots
\end{array}$$

We define the *mapping cone* $\mathcal{M}_\bullet^\phi$ to be the complex such that $\mathcal{M}_i^\phi = B_i \bigoplus A_{i-1}$, where the differential takes $b_i \oplus a_{i-1} \mapsto \left(d_i b_i + (-1)^{i-1}\phi_{i-1}(a_{i-1})\right) \oplus \delta_{i-1}(a_{i-1})$. It is easy to check that $\mathcal{M}_\bullet^\phi$ is a complex. Note that $B_\bullet$ is a subcomplex, and the quotient is $A_{\bullet-1}$ (the complex $A_\bullet$, but with the indices shifted so that the degree i term is A_{i-1}). It is straightforward to check that the mapping cone is a complex.

It is also straightforward to check that $\mathcal{K}_\bullet(x_1, \ldots, x_d; M)$ is the mapping cone of the map

$$\phi_\bullet : \mathcal{K}_\bullet(x_1, \ldots, x_{d-1}; M) \to \mathcal{K}_\bullet(x_1, \ldots, x_{d-1}; M)$$

given by multiplication by x_d on each module.

We next observe that if

$$0 \to A_\bullet \xrightarrow{\phi} B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of complexes, then the homology of the mapping cone $\mathcal{M}_\bullet^\phi$ of $A_\bullet \hookrightarrow B_\bullet$ is the same as $H_\bullet(C_\bullet)$. The isomorphism is induced by

$$\mathcal{M}_i^\phi = B_i \oplus A_{i-1} \twoheadrightarrow B_i \twoheadrightarrow C_i.$$

For a suitable choice of $\pm$, $u \oplus v$ is a cycle in $\mathcal{M}_n^\phi$ iff $du = \pm\phi(v)$. Note that we automatically have $\delta(v) = 0$, since $\phi(\delta(v)) = d(\phi(v)) = \pm ddu = 0$, and ϕ is injective. The cycle is completely determined by u, and u occurs in a cycle iff its image represents a cycle in C_i. The module of boundaries is $d(B_{i+1}) + \phi(A_i) \subseteq B_i$, and obviously maps onto the module of boundaries in C_i. $\square$

Corollary. *If x_d is not a zerodivisor on M, then*

$$H_i(x_1, \ldots, x_d; M) \cong H_i(x_1, \ldots, x_{d-1}; M/x_d M)$$

for all i.

Proof. We apply the discussion of mapping cones when the map ϕ is injective with $A_\bullet = B_\bullet = \mathcal{K}_\bullet(x_1, \ldots, x_{d-1}, M)$, and $\phi = \cdot x_n$. The fact that x_n is not a zerodivisor on M implies that the map ϕ is injective. Note that $C_\bullet \cong \mathcal{K}_\bullet(x_1, \ldots, x_{d-1}; M/x_n M)$. The stated result is immediate.

Theorem. *Let (R, m, K) be local of dimension d and let $x \in m$ be part of a system of parameters generating a reduction of m. Suppose that x is not a zerodivisor on M. Then $e(M) = e(M/xM)$, where M/xM is viewed as a module over R/xR.*

Proof. Let $x_1, \ldots, x_d$ be a system of parameters generating a reduction I of m, where $x = x_n$. Then the images of $x_1, \ldots, x_{d-1}$ generate a reduction J for m/xR in R/xR. Thus, $e(M) = e_I(M)$, and $e(M/xM) = e_J(M/xM)$, and we may compute each of these as an alternating sum of lengths of Koszul homology. But the correspondingly indexed Koszul homology modules are isomorphic by the preceding Corollary. $\square$

Our next goal is to prove that, under mild conditions, rings of multiplicity 1 are regular. We first need:

Lemma (Hironaka). *Let (R, m, K) be a local domain and let $x \in R - \{0\}$ be such that xR has a unique minimal prime P. Suppose that R/P is normal and that R_P is a discrete valuation ring in which x generates the maximal ideal PR_P. Suppose also that the normalization S of R is module-finite over R (which is true when R is complete) and that every minimal prime Q of xS lies over P (which is true if R is universally catenary). Then R is normal, and $P = xR$.*

Proof. Note that if R is universally catenary, and Q is any minimal prime of xS in S, if P' is the contraction of Q to R, the height of P' must be one by the dimension formula: R and S have the same fraction field, and $R/P' \hookrightarrow S/Q$ is module-finite, so that the extension or residue class fields $R_{P'} \to S_Q$ is algebraic. Since P' contains x, we must have $P' = P$.

Since R_P is a discrete valuation ring, it is normal and so $S \subseteq R_P$. Hence, $S_P = R_P$ is already local of dimension one, and S_Q is a further localization of dimension one. It follows that $S_Q = R_P$, and that $QS_Q = PR_P$. Moreover, since $QS_Q \cap S = Q$, we have that $PR_P \cap S = Q$, and so only one prime Q of S lies over P.

We have that S/Q is contained in the fraction field of R/P, and it is an integral extension. Since R/P is normal, we must have that $S/Q = R/P$, and so every residue class in S/Q can be represented by an element of R. This implies that $S = Q + R$. We can also see that $xS = Q$: we have that $xS \subseteq Q$, and to check $Q \subseteq xS$ it suffices to show that this is true after localization at each minimal prime of xS, since S is normal. Q is the only such prime, and $QS_Q = PR_P = xR_P = xS_Q$. Since $S = Q + R$, we now have that $S = xS + R$. By Nakayama's lemma, this implies that $S = R$, so that R is normal. Then $P = Q$, and $Q = xS = xR$. $\square$

We shall say that a module M over a Noetherian ring R *has pure dimension d* (M may be equal to R) if for every associated prime P of M, $\dim(R/P) = d$. An equivalent condition is that every nonzero submodule of M has dimension d.

Theorem. *Let (R, m, K) by a local ring. The R is regular if and only if R has multiplicity 1 and suppose $\widehat{R}$ has pure dimension.*

Proof. If R is regular, it is Cohen-Macaulay and its multiplicity is the length when w we kill a regular system of parameters, which is 1. Moreover, $\widehat{(R)}$ is again regular, and so is a domain. We therefore only need to show the "If" part: we assume that R has multiplicity 1 and $\widehat{R}$ is of pure dimension, and we need to prove that R is regular.

We use induction on $\dim(R)$. If $\dim(R) = 0$, then $e(R) = \ell(R) = 1$, so that R must be a field and is regular.

We may replace R by $\widehat{R}$ without affecting any relevant issue. Then

$$(*) \quad e(R) = \sum_P \ell(R_P) e(R/P)$$

where P runs through all the associated primes of R, each of which is minimal and such that $\dim(R/P) = \dim(R)$, by hypothesis. It follows that there is only one associated

prime, necessarily minimal, and that R_P has length one, and so is a field. This implies that R is a domain.

If the residue field of R is infinite, we can complete the argument as follows. Choose $x \in R$ so that it is part of system of parameters that generates a reduction of m. If $\dim(R) = 1$, the $e(R) = \ell(R/xR) = 1$, so that R/xR is a field and $m = xR$, which shows that R is regular.

If $\dim(R) \geq 2$, then we still have $e(R/xR) = e(R) = 1$. Thus, applying $(*)$ of the second paragraph to R/xR, we find that xR has a unique minimal prime P in R (*a priori*, xR may have embedded primes), that $(R/xR)_P$ is a field, so that $PR_P = xR_P$, and that $e(R/P) = 1$. By the induction hypothesis, R/P is regular, and, therefore, normal. R is universally catenary (complete local rings are homomorphic images of regular rings) and and has a module-finite normalization. Hence, we are in the situation of Hironaka's Lemma, and $P = xR$. Since R/xR is regular, so is R.

If the residue field of R is finite, we may replace R by $R(t)$. The theory of excellent rings then implies that if we complete again, the hypothesis we need on associated primes is preserved: the completion of a ring of ring or module of pure dimension has pure dimension in the excellent case. (One can reduce this to studying the situation when R is an excellent local domain. The theory of excellent rings then yields that the completion is reduced, and is such that all minimal primes have quotients of the same dimension.)

An alternative is to take an irreducible polynomial f of large degree over the residue field K of R, lift it to a monic polynomial g over R, and replace R by $R_1 = R[x]/(g)$. This ring is still complete, and it is module-finite and free over R, so that it has pure dimension. Killing m gives $L = K[x]/(f)$, which is a field. Therefore the new ring still has multiplicity one. Once the cardinality of L is sufficiently large, there will exist a system of parameters of R_1 that gives a reduction of the maximal ideal of R_1, and we can proceed as above. □