Math 711: Lecture of October 30, 2006

Examples. Let $R = K[[x,\, y]]/(x^2, xy)$. This ring has a unique minimal prime, xR, and $m = (x,\, y)R$ is embedded. The image $\overline{x}$ of x in the ring generates a submodule isomorphic to R/m, which has lower dimension. Then $e(R) = e(R/xR) = e(K[[y]]) = 1$.

Likewise, if $R = K[[x,\, y,\, z]]/\bigl((x, y) \cap (z)\bigr)$, then R has two minimal primes, $(x, y)R$ and zR. Thus, $\dim(R) = \dim(R/zR) = \dim(K[[x,\, y]]$, while the module $zR \cong R/(x, y) \cong K[[z]]$ is one-dimensional. Thus, $e(R) = e(R/zR) = e(K[[x,\, y]]) = 1$.

These examples illustrate that a local ring of multiplicity 1 need not be regular. In the first example, R_{red} is a domain. In the second, R is reduced, but not equidimensional.

Finally, consider $R = K[[u,\, v,\, x,\, y,\, z]]/\bigl((u, v) \cap (x, y) \cap z\bigr)$. This ring is reduced but not equidimensional. It has dimension 4 (when we kill zR we get $K[[u,\, v,\, x,\, y]]$), but has two minimal primes with quotients of dimension 3. Consider the ring obtained when we localize at $P = (u,\, v,\, x,\, y)$. The localization S of $T = K[[u,\, v,\, x\,y,\, z]]$ at $(u,\, v,\, x,\, y)T$ is regular of dimension 4, and $u,\, v,\, x,\, y$ is a regular system of parameters. Thus, $R_P = S/\bigl((u,\, v) \cap (x,\, y)\bigr)$ has two minimal primes with quotients that are regular of dimension 2. It follows that $e(R) = 1$ while $e(R_P) = 2$. The problem here is that we "localized away" the relevant minimal prime of R that governed its multiplicitiy.

Discussion: localization. One expects that under mild conditions, $e(R_P) \leq e(R)$. But we only expect this for primes P such that $\dim(R_P) + \dim(R_P) = \dim(R)$. (We always have $\dim(R/P) + \dim(R_P) \leq \dim(R)$. The condition of equality means that P is part of a chain of primes of maximum length, $\dim(R)$, in R.) It is conjectured that in all local rings, whenever $\dim(R_P) + \dim(R_P) = \dim(R)$, one has that $e(R_P) \leq e(R)$.

In studying this problem, one is naturally led to Lech's Conjecture. The result on localization is true if R is excellent (and under various weaker hypotheses), but, so far as I know, remains open in the general case. It would follow, however, from a proof of Lech's Conjecture, which permits a reduction to the case where the ring is complete.

First note:

Lemma. *Let P be a prime ideal of a local ring R. Then:*

(a) *For every minimal prime Q of $P\widehat{R}$,* $\mathrm{height}\,(Q) = \mathrm{height}\,(P)$.

(b) *If* $\dim(R/P) + \dim(R_P) = \dim(R)$, *then there exists a minimal prime Q of PR such that* $\dim(\widehat{R}/Q) + \dim(\widehat{R}_Q) = \dim(\widehat{R})$.

(c) *If $\widehat{R/P}$ is reduced, then with Q as in part (b) we have that $e(R_P) = e(\widehat{R}_Q)$.*

Proof. (a) $R_P \to \widehat{R}_Q$ is faithfully flat, so that $\dim(\widehat{R}_Q) \geq \dim(R_P)$. The minimality of Q implies that PR_P expands to a $Q\widehat{R}_Q$-primary ideal in $\widehat{R}_Q$, so that a system of parameters for R_P will be a system of parameters for $\widehat{R}_Q$ as well.

For (b), note that the completion of R/P, which is $\widehat{R}/P\widehat{R}$, has the same dimension as R/P, and so has a minimal prime, say $Q/P\widehat{R}$, where Q is prime in $\widehat{R}$, such that $\dim(\widehat{R}/Q) = \dim(\widehat{R}/P\widehat{R} = \dim(R/P)$. By part (a), $\dim(\widehat{R}_Q = \dim(R_P)$ as well.

To prove (c), observe that if $\widehat{R/P}$ is reduced, then so is $\widehat{R}_Q/P\widehat{R}_Q$, which means that PR_P expands to the maximal ideal in $\widehat{R}_Q$. The equality of multiplicities then follows from the Proposition on p. 6 of the Lecture Notes of October 20. $\square$

Our next objective, which will take a while, is to prove the following:

Theorem (localization theorem for multiplicities). *If P is a prime ideal of a complete local ring R such that $\dim(R/P) + \dim(R_P) = \dim(R)$, then $e(R_P) \le e(R)$.*

Assuming this for the moment, we have several corollaries.

Corollary. *If P is a prime of a local ring R such that $\dim(R/P) + \dim(R_P) = \dim(R)$ and the completion of R/P is reduced,[1] then $e(R_P) \le e(R)$.*

Proof. Choose a minimal prime Q of $P\widehat{R}$ such that $\dim(\widehat{R}/Q) + \dim(\widehat{R}_Q) = \dim(\widehat{R})$, as in part (b) of the Lemma. Then by part (c),

$$e(R_P) = e(\widehat{R}_Q) \le e(\widehat{R}) = e(R).$$

Corollary. *If Lech's conjecture holds, then for every prime P of a local ring R such that $\dim(R/P) + \dim(R_P) = \dim(R)$, $e(R_P) \le e(R)$.*

Proof. Choose Q as in part (b) of the Lemma. Then $R_P \to \widehat{R}_Q$ is flat local, and so by Lech's conjecture

$$e(R_P) \le e(\widehat{R}_Q) \le e(\widehat{R}) = e(R). \qquad \square$$

We also get corresponding results for modules.

Corollary. *If R is a local ring, M a finitely generated R-module, and P is a prime of the support of M such that $\dim(R/P) + \dim(M_P) = \dim(M)$, then:*

(a) *If the completion of R/P is reduced, then $e(M_P) \le e(M)$.*

(b) *If Lech's conjecture holds, then $e(M_P) \le e(M)$.*

Proof. Note that we can replace R by $R/\mathrm{Ann}_R M$, so that we may assume that M is faithful and $\dim(R) = \dim(M) = d$, say. Note that M is faithful if and only if for some (equivalently, every) finite set of generators $u_1, \ldots, u_h$ for M, the map $R \to M^{\oplus h}$ such

[1] This is always true if R is excellent: the completion of an excellent reduced local ring is reduced.

that $r \mapsto (ru_1, \ldots, ru_h)$ is injective. This condition is obviously preserved by localization. Now,

$$(*) \quad e(M) = \sum_{1 \le i \le h,\, \dim(R/P_i) = d} \ell_{R_{P_i}}(M_{P_i}) e(R/P_i).$$

Note that once we have that M is faithful, $\dim(R/P) + \dim(M_P) = \dim(M)$ is equivalent to $\dim(R/P) + \dim(R_P) = \dim(R)$, since M_P is faithful over R_P. The minimal primes of M and R are the same, and so are the minimal primes of M_P and R_P: the latter correspond to the minimal primes of R that are contained in P. There is a formula like $(*)$ for $e(M_P)$, where the summation is extended over minimal primes $\mathfrak{p}$ of the support of M_P, i.e., of R_P, such that $\dim(R_P)/\mathfrak{p} = \dim(M_P)$, which is $\dim(R_P)$. Let $\mathfrak{p}$ be such a minimal prime. Then there is a chain of primes from $\mathfrak{p}$ to P of length height (P), and this can be concatenated with a chain of primes of length $\dim(R/P)$ from P to m, producing a chain of length $\dim(R)$. It follows that $\dim(R/\mathfrak{p}) \ge d$, and the other inequality is obvious. Therefore, $\mathfrak{p}$ is one of the P_i. Moreover, in R/P_i, we still have

$$\dim\big((R/P_i)/(P/P_i)\big) + \operatorname{height}\big((R/P_i)_{P/P_i}\big) = \dim(R/P_i) = d.$$

Thus, the terms in the formula corresponding to $(*)$ for M_P correspond to a subset of the the terms occurring in $(*)$, but have the form

$$\ell_{R_{P_i}}(M_{P_i}) e(R_P/P_iR_P).$$

Note that each P_i occurring is contained in P, and localizing first at P and then at P_iR_P produces the same result as localizing at P_i. Using either (a) or (b), whichever holds, we have that every $e\big((R/P_i)_P\big) \le e(R/P_i)$. $\square$

We next want to understand multiplicities in the hypersurface case.

Theorem. *Let (R, m, K) be a regular local ring of dimension d and let $f \in m$. Let $S = R/fR$. The $e(S)$ is the m-adic order of f, i.e., the unique integer k such that $f \in m^k - m^{k+1}$.*

Proof. We use induction on $\dim(R)$. If $\dim(R) = 1$ the result is obvious. Suppose $\dim(R) > 1$. We replace R by $R(t)$ if necessary so that we may assume the residue class field is infinite. Choose a regular system of parameters $x_1, \ldots, x_d$ for R. By replacing these by linearly independent linear combinatons we may assume that x_1 is such that

(1) x_1 does not divide f, so that the image of x_1 is not a zerodivisor in S.

(2) The image of x_1 in m/m^2 does not divide the leading form of f in $\mathrm{gr}_m(R)$.

(3) The image of x_1 in S is part of a minimal set of generators for a minimal reduction of m/fR, the maximal ideal of S.

Let $\overline{x}$ be the image of x_1 in S. Then $e(S) = e(S/\overline{x}S)$, and this is the quotient of the regular ring R/x_1R by the image of f. Moreover, the (m/x_1R)-adic order of the image of

f in R/x_1R is the same as the m-adic order of f in R. The result now follows from the induction hypothesis applied to the image of f in R/x_1R. $\square$

We next want to reduce the problem of proving the localization result for complete local domains to proving the following statement:

Theorem (symbolic powers in regular rings). *Let $P \subseteq Q$ be prime ideals of a regular ring R. Then $P^{(n)} \subseteq Q^{(n)}$ for every positive integer n.*

We postpone the proof for the moment. Note, however, that one can reduce at once to the local case, where Q is the maximal ideal, by working with $(R_Q,\, QR_Q)$ instead of R.

Discussion: the symbolic power theorem for regular rings implies that multiplciities do not increase under localization. Let R be complete local, and let P be a prime ideal of R such that $\dim(R/P) + \mathrm{height}(R_P) = \dim(R)$. We want to show that $e(R_P) \leq e(R)$. Exactly as in the discussion of the module case in the proof of the Corollary, one can reduce to the case where R is a domain. As usual, one may assume without loss of generality that the residue field is sufficiently large for R to have a system of parameters $x_1, \ldots, x_d$ that generates a minimal reduction of m. Then in the equicharacteristic case (respectively, the mixed characteristic case), we can map $K[[X_1, \ldots, X_d]] \to R$ (respectively, $V[[X_1, \ldots, X_d]] \to R$), where $K \subseteq R$ (respectively $V \subseteq R$) is a coefficient field (respectively, a complete DVR that is a coefficient ring) and so that $X_i \mapsto x_i$, $1 \leq i \leq d$. In both cases, R is module-finite over the image A: in the equicharacteristic case, $A = K[[x_1, \ldots, x_d]]$ is regular, while in mixed characteristic the kernel of $V[[X_1, \ldots, X_d]] \to R$ must be a height one prime, and therefore principle, so that $A \cong V[[x_1, \ldots, x_d]]/(f)$. Since the maximal ideal of R is integral over $(x_1, \ldots, x_d)R$ and R is module-finite over A, the maximal ideal of A is also integral over $(x_1, \ldots, x_d)A$. Let ρ denote the torsion-free rank of R as an A-module, which is the same as the degree of the extension of fraction fields. Suppose that P is a prime of R and let $\mathfrak{p}$ be its contraction to A. Let I be the ideal $(x_1, \ldots, x_d)A$. Then $e(R) = e_{IR}(R)$, which is the same as $e_I(R)$ with R thought of as an A-module. This is $\rho\, e_I(A) = \rho\, e(A)$. The result on symbolic powers gives the result on localization of multiplicities for $A = T/(f)$, when T is regular: one multiplicity is the order of f in T with respect to the maximal ideal, while the other is the order of f in a localization of T. (In the equicharacteristic case, both A and its localization are regular, and both multiplicities are 1.) Thus, $\rho\, e(A_\mathfrak{p}) \leq \rho\, e(A) = e(R)$. But we shall see in the sequel that $e(R_P) \leq e_\mathfrak{p}(R_\mathfrak{p})$, with $R_\mathfrak{p}$ is viewed as an $A_\mathfrak{p}$ module. Since R is module-finite over A, $R_\mathfrak{p}$ is module-finite over $A_\mathfrak{p}$, and $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$ is an Artin ring, and is a product of local rings one of which is $R_P/(\mathfrak{p}^n R_P)$. Then

$$\ell_{A_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}) \geq \ell_{A_\mathfrak{p}}(R_P/\mathfrak{p}^n R_P) \geq \ell_{A_\mathfrak{p}}(R_P/P^n R_P) \geq \ell_{R_P}(R_P/P^n R_P)$$

for all n, so that the multiplicity of $R_\mathfrak{p}$ as an $A_\mathfrak{p}$-module is greater than or equal to $e(R_P)$. But then

$$e(R_P) \leq e_\mathfrak{p}(R_\mathfrak{p}) = \rho\, e(A_\mathfrak{p}) \leq \rho\, e(A) = e(R),$$

as required. $\square$

Thus, all that remains is to prove the theorem on symbolic powers in regular rings.