Math 711: Lecture of November 1, 2006

Before attacking the problem of comparing symbolic powers of primes, we want to discuss some techniques that will be needed. One is connected with enlarging the residue class field of a local ring.

Proposition. *Let $(R,\, m,\, K)$ be a local ring, and let θ be an element of the algebraic closure of K with minimal monic irreducible polynomial $f(x) \in K[x]$. Let $F(x)$ be a monic polynomial of the same degree d as f that lifts F to $R[x]$. Let $S = R[x]/(F)$. Then S is module-finite, free of rank d, and local over R. Hence, S is R-flat. The residue field of S is isomorphic with $L = K[\theta]$, and S has maximal ideal mS.*

Proof. S is module-finite and free of rank d over R by the division algorithm. Hence, every maximal ideal of S must lie over m, and the maximal ideals of S correspond bijectively to those of $S/mS = R[x]/(mR[x] + FR[x]) \cong K[x]/fK[x] \cong K[\theta]$, which shows that mS is maximal and that it is the only maximal ideal of S. This also shows that $S/mS \cong K[\theta]$. $\square$

Discussion: getting reductions such that the number of generators is the analytic spread. Let $(R,\, m,\, K)$ be local and I an ideal with analytic spread h. One way of enlarging the residue field so as to guarantee the existence of a reduction of I with h generators is to replace R by $R(t)$, so that the residue class field becomes infinite. For this purpose, it is not necessary to enlarge R so that K becomes infinite. One only needs that K have sufficiently large cardinality. When K is finite, one can choose a primitive element θ for a larger finite field extension L: the cardinality of the finite field L may be taken a large as one likes, and a primitive element exists because the extension is separable. Recall that the issue is to give one-forms of $B = K \otimes_R \mathrm{gr}_I(R)$ that are a homogeneous system of parameters. After making the type of extension in the Proposition, one has, because mS is the maximal ideal of S, that

$$L \otimes_S \mathrm{gr}_{IS}(S) \cong L \otimes_S \big(S \otimes_R \mathrm{gr}_I(R)\big) \cong L \otimes_R \mathrm{gr}_I(R) \cong L \otimes_K \big(K \otimes_R \mathrm{gr}_I(R)\big).$$

If one makes a base change to $\overline{K} \otimes_K B$, where $\overline{K}$ is the algebraic closure of K, one certainly has a linear homogeneous system of parameters. The coefficients will lie in L for any sufficiently large choice of finite field L.

Proposition. *Let $(R,\, m,\, K)$ be any complete local ring. Then R has a faithfully flat extension $(S,\, \mathfrak{n}, L)$ such that $\mathfrak{n} = mS$ and L is the algebraic closure of K. If R is regular, then S is regular.*

Proof. We may take R to be a homomorphic image of $T = K[[x_1,\, \ldots\, , x_d]]$, where K is a field, or of $T = V[[x_1,\, \ldots\, , x_d]]$, where $(V,\, \pi V, K)$ is a complete DVR such that the induced map of residue class fields is an isomorphism. In the first case, let L be the algebraic closure of K. Then $T_1 = L[[x_1,\, \ldots\, , x_n]]$ is faithfully flat over T, and the expansion of

$(x_1, \ldots, x_d)T$ to T_1 is the maximal ideal of T_1. Here , faithful flatness follows using the Lemma on p. 2 of the Lecture Notes of October 18, becasuse every system of parameters for T is a system of parameters for T_1, and so a regular sequence on T_1, since T_1 is Cohen-Macaulay. Then $S = T_1 \otimes_T R$ is faithfully flat over R, has residue class field L, and m expands to the maximal ideal.

We can solve the problem in the same way in mixed characteristic provided that we can solve the problem for V: if $(W, \pi W, L)$ is a complete DVR that is a local extension of V with residue class field L, then $T_1 = W[[x_1, \ldots, x_d]]$ will solve the problem for T, and $T_1 \otimes_T R$ will solve the problem for R, just as above.

We have therefore reduced to studying the case where the ring is a complete DVR V. Furthermore, if $(W, \pi W, L)$ solves the problem but is not necessarily complete, we may use $\widehat{W}$ to give a solution that is a complete DVR.

Next note that if $(V_\lambda, \pi V_\lambda, K_\lambda)$ is a direct limit system of DVRs, all with the same generator π for their maximal ideals, such that the maps are local and injective, then $\varinjlim_\lambda V_\lambda$ is DVR with maximal ideal generated by π. It is then clear that the residue class field is $\varinjlim_\lambda K_\lambda$. The reason is that every nonzero element of the direct limit may be viewed as arising from some V_λ, and in that ring it may be written as a unit times a power of π. Thus, every nonzero element of the direct limit is a unit times a power of π.

We now construct the required DVR as a direct limit of DVRs, where the index set is given by a well-ordering of the field L, the algebraic closure of K, in which 0 is the least element. We shall construct the family $\{(V_\lambda, \pi, , K_\lambda)\}_{\lambda \in L}$ in such a way that for every $\lambda \in L$,

$$\{\mu \in L : \mu \leq \lambda\} \subseteq K_\lambda \subseteq L.$$

This will complete the proof, since the direct limit of the family will be the required DVR with residue class field L.

Take $V_0 = V$. If $\lambda \in L$ and V_μ has been constructed for $\mu < \lambda$ such that for all $\mu < \lambda$,

$$\{\nu \in L : \nu \leq \mu\} \subseteq K_\mu \subseteq L,$$

then we proceed as follows to construct V_λ. There are two cases.

(1) If λ has an immediate predecessor μ and $\lambda \in K_\mu$ we simply let $V_\lambda = V_\mu$, while if $\lambda \notin K_\mu$, we take $\theta = \lambda$ in the first Proposition to construct V_λ.

(2) If λ is a limit ordinal, we first let $(V', \pi V', K') = \varinjlim_{\mu < \lambda} V_\mu$. If λ is in the residue class field of V', we let $V_\lambda = V'$. If not, we use the first Proposition to extend V' so that its residue class field is $K'[\lambda]$. $\square$

To prove the theorem on comparison of symbolic powers in regular rings, we shall also need some results on valuation domains that are not necessarily Noetherian. In particular, we need the following method of constructing such valuation domains.

Proposition. *Let $(V, \mathfrak{n}, L)$ be a valuation domain with fraction field $\mathcal{K}$ and let (W, m, K) be a valuation domain with fraction field L. Let $g : V \twoheadrightarrow L$ be the quotient map. Then $T = \{v \in V : g(v) \in W\} \subseteq V$ is a valuation domain with fraction field $\mathcal{K}$. Its maximal ideal is $\{v \in V : g(v) \in m\}$. Its residue class field is K, and it contains a prime ideal $\mathfrak{q}$ which may be described as $\mathfrak{n} \cap T$. Moreover $T/\mathfrak{q} = W$, while $T_{\mathfrak{q}} = V$.*

Proof. Let $f \in \mathcal{K}$ be nonzero. If $f \notin V$ then $1/f$ is not only in V: it must be in $\mathfrak{n}$, and so has image 0 in L. Thus, $1/f \in T$. If $f \in V - \mathfrak{n}$ then $1/f \in V - \mathfrak{n}$ as well. The images of these two elements are reciprocals in $W/\mathfrak{n} = K$, and so at least one of the two is in W. Thus, either f or $1/f$ is in V. Finally, if $f \in \mathfrak{n}$ then $g(f) = 0 \in W$, and so $f \in T$. This shows that V is a valuation domain with fraction field $\mathcal{K}$.

The restriction of g to T clearly maps T onto K. This means that the kernel of this map must be the unique maximal ideal of T, and that the residue class field is K. The prime $\mathfrak{q}$ is clearly the kernel of the surjection $T \twoheadrightarrow W$ obtained by restricting g to T, whence $T/\mathfrak{q} = W$. Since $\mathfrak{n}$ lies over $\mathfrak{q}$, we have an induced local map $T_{\mathfrak{q}} \to V$ of valuation domains of $\mathcal{K}$. This map must be the identity by the third Remark on p. 1 of the Lecture Notes of October 2. $\square$

The valuation domain T is called the *composite* of V and W.

Corollary. *Let R be a domain with fraction field $\mathcal{K}$, and*

$$P_0 \subseteq P_1 \subseteq \cdots \subseteq P_k$$

a chain of prime ideals of R. Then there exists a valuation domain V with $R \subseteq V \subseteq \mathcal{K}$ and a chain of prime ideals

$$\mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_k$$

of V such that $\mathfrak{q}_i \cap R = P_i$, $1 \le i \le k$. Morevover, we may assume that $\mathfrak{q}_k$ is the maximal ideal of V.

Proof. If $n = 0$, we simply want to find V a valuation domain with maximal ideal $\mathfrak{q}$ lying over $P = P_0$. We may replace R by R_P and apply the Corollary on p. 2 of the Lecture Notes of September 11 with $I = PR_P$ and $L = \mathcal{K}$.

Now suppose that V_{k-1} together with

$$\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_{k-1}$$

solve the problem for

$$P_0 \subseteq \cdots \subseteq P_{k-1}.$$

If $P_k = P_{k-1}$ take $V = V_{k-1}$ and $\mathfrak{q}_k = \mathfrak{q}_{k-1}$. If $P_{k-1} \subset P_k$ is strict, we can choose a valuation domain W of the fraction field of R/P_{k-1} containing R/P_{k-1} and whose maximal ideal lies over P_k/P_{k-1}. Take V to be the composite of V_{k-1} and W. $\square$