Math 711: Lecture of November 6, 2006

To finish our comparison of symbolic powers in a regular ring, we shall make use of quadratic transforms (also called *quadratic transformations* or *quadratic dilatations*) in a more general context than in the proof of the Lipman-Sathaye Jacobian Theorem.

Let (R, m, K) be a local domain with $R \subseteq (V, \mathfrak{n})$ a local map, where V is a not necessarily Noetherian valuation domain. The *first quadratic transform* of R along V is the localization (R_1, m_1) of $R[m/x]$ at the contraction of $\mathfrak{n}$, where x is any element of m such that $xV = mV$. This ring is again a local ring with a local map $R_1 \to V$.

The quadratic transform is independent of the choice of the element x. To see this, suppose that $xV = yV$, where $y, x \in m$. Then $y/x \in R[m/x]$ is a unit in V, so its inverse $x/y \in R_1$. Since $m/y = (m/x)(x/y)$, it follows that $R[m, y] \subseteq R_1$. Moreover, each element of $R[m/y]$ that is invertible in V has an inverse in R_1, so that if Q is the contraction of $\mathfrak{n}$ to $R[m/y]$ we have an induced inclusion map $R[m/y]_Q \to R_1$. An exactly symmetric argument gives the opposite inclusion.

As in our earlier situation, we may take iterated quadratic transforms

$$R \subseteq R_1 \subseteq \cdots \subseteq R_k \subseteq \cdots \subseteq V.$$

Note that if $m = x_1, \ldots, x_h$, then $mV = (x_1, \ldots, x_h)V$, so that x may be chosen from among the x_i. Putting this together with the Lemma on p. 2 of the Lecture Notes of September 29, we have:

Proposition. *Let (R, m, K) be regular local with $x_1, \ldots, x_d$ a regular system of parameters and suppose that $R \subseteq (V, \mathfrak{n})$ is local where V is a valuation domain. If the x_i are numbered so that $x_jV \subseteq x_1V$ for all $j > 1$, then the quadratic transform R_1 is a localization of the ring $S = R[x_2/x_1, \ldots, x_d/x_1]$, which is regular of dimension d. In particular, R_1 has dimension at most d. Moreover, $S/x_1S \cong K[X_2, \ldots, X_d]$, where X_i is the image of x_i/x_1, $2 \le i \le d$.* $\square$

Here is another important example:

Theorem. *Let R be a one dimensional local domain whose integral closure $(V, \mathfrak{n})$ is local and module-finite over R. (This is always the case if R is a complete one-dimensional local domain.) Let*

$$R \subseteq R_1 \subseteq \cdots \subseteq R_k \subseteq \cdots \subseteq V$$

be the sequence of iterated quadratic transforms. Then for all sufficiently large k, $R_k = V$.

Proof. Since V is module-finite over R, it cannot have an infinite ascending chain of R-submodules. It follows that the chain R_i is eventually stable. But if the maximal ideal of

R_i is not principal and has minimal generators $y_1, \ldots, y_h$ with y_1 of least order in V, then for some $j > 1$, $y_j/y_1 \in V - R_i$, and $y_j/y_1 \in R_{i+1}$. Therefore, for sufficiently large i, the maximal ideal of R_i is prinicpal. But then R_i is a DVR, and is a normal ring inside the fraction field of R and containing R. It follows that $R_i = V$. $\square$

We also note:

Theorem. *Let (R, m, K) be a local domain with $R \subseteq (V, \mathfrak{n})$ a local inclusion, where V is a valuation domain of the fraction field of R. Let $\mathfrak{q}$ be a prime ideal of V lying over $P \neq m$ in R. Let*

$$R \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_k \subseteq \cdots \subseteq V$$

be the sequence of quadratic trransforms of R along V. Let P_i be the contraction of P to R_i. Then

$$R/P \subseteq R_1/P_1 \subseteq R_2/P_2 \subseteq \cdots \subseteq R_k/P_k \subseteq \cdots \subseteq V/\mathfrak{q}$$

is the sequence of quadratic transforms of $R/P \subseteq V/\mathfrak{q}$ along $V/\mathfrak{q}$.

Proof. By induction on k, it suffices to see this when $k = 1$. Let $x_1, \ldots, x_h$ generate the maximal ideal m of R with $x_1V = mV$. Some $x \in m$ is not in $\mathfrak{n}$, and since $xV \subseteq mV = x_1V$, $x_1 \notin \mathfrak{q}$ and so $x_1 \notin P$. Moreover, $x_1(V/\mathfrak{q}) = (m/P)(V/\mathfrak{q})$. It follows that the quadratic transform of R/P along $V/\mathfrak{q}$ is the localization at the contraction of $\mathfrak{n}$ of $(R/P)[\widetilde{m}/\overline{x}_1]$, where $\widetilde{m}$ is m/P and $\overline{x}_1$ is the image of x_1 in R/P. The stated result follows at once. Note that we again have $P_1 \neq m_1$, the maximal ideal of R_1, since $x_1 \in m_1 - P_1$. $\square$

We next observe:

Lemma. *Let (R, m, K) be a regular local ring with algebraically closed residue class field, and suppose $R \subseteq (V, \mathfrak{n})$ is local, where V is a valuation domain and $R/m \to Vn$ is an isomorphism. Then there is a regular system of parameters $x_1, \ldots, x_d$ for R such that the first quadratic transform is the localization of $R[x_2/x_1, \ldots, x_d/x_1]$ at the height d maximal ideal generated by $x_1, x_2/x_1, \ldots, x_d/x_1$, so that these elements are a regular system of parameters in the first quadratic transform.*

Proof. Let $x_1, y_2, \ldots, y_d$ be one regular system of parameters for R such that $x_1V = mV$. $\mathfrak{n}$ contains x_1, and so $\mathfrak{n}$ lies over a prime ideal of R containing x_1. Hence, the quotient of $R[m/x_1]$ by the contraction of $\mathfrak{n}$ is also a quotient of $K[Y_2, \ldots, Y_d]$, where Y_i is the image of y_i/x_1, $2 \leq i \leq d$. The resulting quotient domain imbeds embeds K-isomorphically in $K = V/\mathfrak{n}$, and so is equal to K. It follows that the contraction of $\mathfrak{n}$ corresponds to a maximal ideal of $K[Y_2, \ldots, Y_d]$, which must have the form $(Y_2 - c_2, \ldots, Y_d - c_d)$ for elements $c_2, \ldots, c_d \in K$. Therefore we may let $x_i = y_i - c_ix_1$ for each i, $2 \leq i \leq d$. $\square$

Proof of the theorem on comparison of symbolic powers. We want to show that if R is regular and $P \subseteq Q$ are prime, then $P^{(n)} \subseteq Q^{(n)}$ for all n. By considering a saturated chain of primes joining P to Q we immediately reduce to the case where the height if Q/P in

R/P is one. We may replace R by R_Q, and so we may assume that Q is m in the regular local ring (R, m) and that $\dim(R/P) = 1$.

Suppose that $(R,\, m,\, K) \to S$ is a flat local map, where S is regular with maximal ideal mS. Then it suffices to prove the theorem for S, for if P_1 in S lies over P and we know the theorem for S, we have

$$P^{(n)} \subseteq P_1^{(n)} \subseteq (mS)^n = m^n S,$$

and then

$$P^{(n)} \subseteq m^n S \cap R = m^n,$$

because S is faithfully flat over R. We may therefore replace R first by its completion, and then by a complete regular local ring with an algebraically closed residue field. Hence, from now on, we shall assume that R is complete with residue class field K that is algebraically closed, as well as that $\dim(R/P) = 1$.

We now introduce valuations. Let V_1 be a valuation domain of the fraction field of R whose maximal ideal contracts to P: we may use, for example, order with respect to powers of PR_P to construct V_1. Let W be the integral closure of R/P, which will be a discrete valuation ring because R/P is a complete local domain of dimension one. Since K is algebraically closed, the residue class field of W is K. Let $(V, \mathfrak{n})$ be the composite valuation. Then $\mathfrak{n}$ lies over m, and $V/\mathfrak{n} = K$. Moreover, V has a prime [q] lying over P. Now consider the sequence of quadratic transforms

$$(R,\, m,\, K) \subseteq (R,\, m_1,\, K) \subseteq (R,\, m_2,\, K) \subseteq \cdots \subseteq (R_k,\, m_k,\, K) \subseteq \cdots \subseteq (V,\, \mathfrak{n},\, K).$$

Each R_i has a prime P_i that is the contraction of $\mathfrak{q}$. Now R/P_k is the kth quadratic transform of R/P, by the Theorem above, and so for large k is the DVR W, by the earlier Theorem. Then R_k/P_k is regular, and so P_k is generated by part of a regular system of parameters. We shall see in the sequel that $P_k^{(n} = P_k^n \subseteq m^n$ in this case. Assuming this, to complete the proof it suffices to show that if a given R_i provides a counterexample (where $R_0 = R$), then so does its quadratic transform.

We might as well work with R and

$$R_1 = R[x_2/x_1,\, \ldots\,,\, x_d]_{\mathcal{M}},$$

where $\mathcal{M}$ is the maximal ideal $(x_1,\, x_2/x_1,\, \ldots\,, x_d/x_1)R[m/x_1]$. Suppose $f \in R$ has m-adic order n, but order at least $n+1$ in R_P. Since $m^n/x_1^n \subseteq R[m/x_1]$, we have that $f/x_1^n \in R[m/x_1]$. Since $x_1 \notin P_1$, f_n/x_1^n has the same order as f in $(R_1)_{P_1}$, and since P_1 lies over P this will be at least $n+1$. It therefore will suffice to show that f/x_1^n has m_1-adic order at most n in R_1. Since $R_1/\mathcal{M}^{n+1}$ already local, it suffices to show that $f/x_1^n \notin \mathcal{M}^{n+1}$. Suppose otherwise. The ideal $\mathcal{M}^{n+1}$ is generated by elements μ/x_1^{n+1} where μ is a monomial of degree $n+1$ in $x_1^2, x_2, \ldots, x_n$, and

$$R[m/x_1] = \bigcup_t m^t/x_1^t.$$

Therefore, for some t, we have

$$f/x_1^n \in (1/x_1^{n+1})(x_1^2, x_2, \dots, x_d)^{n+1} m^t / x_1^t$$

and so

$$x_1^{t+1} f \in (x_1^2, x_2, \dots, x_d)^{n+1} m^t.$$

Each of the obvious generators obtained by expanding the product on the right that involves x_1^2 has degree at least $n+t+2$. Hence, in the degree $n+t+1$ part of $\mathrm{gr}_m(R) = K[X_1, \dots, X_d]$, we have that

$$X_1^{t+1} F \in (X_2, \dots, X_d)^{n+1}(X_1, \dots, X_d)^t,$$

where F is the image of f in m^n/m^{n+1}, and is supposedly not 0. By taking homogeneous components in degree $n+t+1$ we see that $x_1^{t+1}F$ must be in the K-vector space span of the obvious monomial generators of

$$(X_2, \dots, X_d)^{n+1}(X_1, \dots, X_d)^t.$$

But this is clearly impossible with $F \neq 0$, since none of these monomials is divisible by X_1^{t+1}.

This completes the proof, once we have shown that for primes generated by a regular sequence, symbolic powers are the same as ordinary powers. $\square$