Math 711: Lecture of November 8, 2006

We have completed the proof of the theorem on comparison of symbolic powers of prime ideals in regular rings as soon as we have established:

Lemma. *Let P be a prime ideal of the ring R that is generated by a regular sequence, $x_1, \ldots, x_k$. Then $P^{(n)} = P^n$ for every integer n.*

Proof. Let $u \in R - P$. We need only show that u is not a zerodivsor on P^n. Suppose $ur \in P^n$ with $r \notin P^n$. Choose h, which may be 0, such that $r \in P^h - P^{h+1}$: evidently, $h < n$. Then $ur \in P^n \subseteq P^{h+1}$. This implies that the image of u in R/P is a zerodivisor on P^h/P^{h+1}. But by part (d) of the Proposition on p. 2 of the Lecture Notes of October 23, P^h/P^{h+1} is a free R/P-module with a free basis in bijective correspondence with monomials of degree h in variables $X_1, \ldots, X_k$. $\square$

Before proceeding further, we want to record an import result on flatness. We first note:

Lemma. *Let M be an R-module with a finite filtration such that $x \in R$ is not a zerodivisor on any factor. Then x is not a zerodivisor on M.*

Proof. By induction on the number of factors, it suffices to consider that case of two factors, i.e., where one has a short exact sequence $0 \to N_1 \to M \to N_2 \to 0$. If $u \in M$ is such that $xu = 0$, then the image of u in N_2 must be 0, or else x will be a zerodivisor on N_2. But then $u \in N_1$, and so $xu = 0$ implies that $u = 0$. $\square$

Next note that when $(R, m, K) \to (S, \mathfrak{n}, L)$ is local and M is an S-module, M/mM is called the *closed fiber* of M (because it is the fiber over the unique closed point m of $\mathrm{Spec}\,(R)$). In this case, if we make a base change to R/I, where $I \subseteq m$ is an ideal of R, R, S, and M become R/I, S/IS, and M/IM, respectively, but the closed fiber does not change: $(M/IM)/m(M/IM) \cong M/mM$.

In the result that follows, the most important case is when $M = S$.

Theorem. *Let $(R, m, K) \to (S, \mathfrak{n}, L)$ be a local homomorphism of local rings and let M be an S-module that is R-flat. Then:*

(a) $\dim(M) = \dim(R) + \dim(M/mM)$.

(b) *If $y \in \mathfrak{n}$ is a nonzerodivisor on M/mM, then it is a nonzerodivisor on M and on M/IM for every ideal $I \subseteq m$ of R. Moreover, if $y \in \mathfrak{n}$ is a nonzerodivisor on M/mM, then M/yM is again flat over R.*

If $\mathrm{depth}_m R = 0$*, then $y \in \mathfrak{n}$ is a nonzerodivisor on M if and only if it is a nonzerodivisor on M/mM.*

(c) $\operatorname{depth}_{\mathfrak{n}} M = \operatorname{depth}_m nR + \operatorname{depth}_{\mathfrak{n}} M/mM$.

Proof. For part (a), we proceed by induction on $\dim(R)$. If $\dim(R) = 0$ then m is nilpotent, and (a) holds even without the assumption that M is R-flat. If $\dim(R) \geq 1$, let $\mathfrak{A}$ be the ideal of nilpotent elements in R, and make a base change to $R/\mathfrak{A}$. The dimensions of R and M do not change, and the closed fiber does not change. Thus, we may assume that R is reduced. But then m contains a nonzerodivisor x, which is consequently also a nonzerodivisor on M because M is R-flat. Make a base change to R/xR. By the induction hypothesis, $\dim(M/xM) = \dim(R/xR) + \dim(M/mM)$. Since $\dim(M/xM) = \dim(M) - 1$ and $\dim(R/xR) = \dim(R) - 1$, the result follows.

For part (b), suppose that y is not a zerodivisor on M/mM. We want to show that y is not a zerodivisor on M/IM. Suppose y kills a nonzero element u of M/IM. We can choose $N \gg 0$ so large that $u \notin m^N(M/IM)$. It follows that y kills the nonzero image of u in $(M/IM)/m^N(M/IM) \cong M/(I + m^N)M$, and so there is no loss of generality in assuming that I is m-primary. In this case, R/I has a finite filtration in which every factor is copy of $K = R/m$. When we apply $M \otimes_R \_$, the fact that M is R-flat implies that M/IM has a finite filtration in which every factor is a copy of $M \otimes R/m \cong M/mM$. By the Lemma above, since y is not a zerodivisor on any of these factors, it is not a zerodivisor on M/IM, as required.

To prove that M' is R-flat, it suffices to show that $\operatorname{Tor}_1^R(N, M') = 0$ for every finitely generated R-module N, since every R-module is a direct limit of finitely generated R-modules. Since a finitely generated R-module N has a finite filtration with cyclic factors, it follows that it suffices to prove that $\operatorname{Tor}^R(R/I, M') = 0$ for every ideal I of R. Let $M' = M/yM$. Starting with the short exact sequence

$$0 \longrightarrow M \xrightarrow{y} M \longrightarrow M/yM \longrightarrow 0$$

we may apply $R/I \otimes_R \_$ to get a long exact sequence part of which is

$$\longrightarrow \operatorname{Tor}_1^R(R/I, M) \longrightarrow \operatorname{Tor}_1^R(R/I, M/yM) \longrightarrow M/IM \xrightarrow{y\cdot} M/IM \longrightarrow \cdots .$$

Since M is R-flat, the leftmost term is 0, and since we have already shown that y is not a zerodivisor on M/IM, it follows that $\operatorname{Tor}_1^R(R/I, M/yM) = 0$ for all I, as required.

We next consider the case where $\operatorname{depth}_m(R) = 0$. Then we can choose a nonzero element $z \in R$ such that $zm = 0$, i.e.,

$$0 \longrightarrow m \longrightarrow R \xrightarrow{z\cdot} R$$

is exact. Applying $\_ \otimes_R M$, we have that

$$0 \longrightarrow m \otimes_R M \longrightarrow M \xrightarrow{z\cdot} M$$

is exact. This shows both that $m \otimes_R M$ may be identified with its image, which is mM, and that $\operatorname{Ann}_M z = mM$. We have already shown that if y is a nonzerodivisor on M/mM then it is a nonzerodivisor on M. For the converse, suppose $u \in M$ is such that $yu \in mM$.

We must show that $u \in mM$. But $zyu \in zmM = 0$, and so $zu = 0$, i.e., $u \in \mathrm{Ann}_M z$, which we have already shown is mM, as required.

To prove part (c), let $x_1, \ldots, x_h \in m$ be a maximal regular sequence in R. Since M is flat, we may make a base change to $R/(x_1, \ldots, x_h)R$, $M/(x_1, \ldots, x_h)M$. Both sides of the equality we are trying to prove decrease by h, since the closed fiber is unchanged. Thus, we may assume without loss of generality that $\mathrm{depth}_m(R) = 0$. We complete the argument by induction on $\dim(M/mM)$. Since $y \in \mathfrak{n}$ is a nonzerodivisor on M/mM if and only if it is a nonzerodivisor on M, if one of these two modules has depth 0 on $\mathfrak{n}$ then so does the other. Therefore, we may assume that $\mathrm{depth}_{\mathfrak{n}} M/mM > 0$. Choose $y \in \mathfrak{n}$ that is a nonzerodivisor on M/mM. Then y is also a nonzerodivisor on M, and M/yM is again R-flat. Let $M/mM = \overline{M}$. We may apply the induction hypothesis to M/yM to conclude that

$$\mathrm{depth}_{\mathfrak{n}}(M/yM) = \mathrm{depth}_{\mathfrak{n}}(\overline{M}/y\overline{M}) + \mathrm{depth}_m(R),$$

since $\overline{M}/y\overline{M}$ may be identified with the closed fiber of M/yM. Since

$$\mathrm{depth}_n(M/yM) = \mathrm{depth}_{\mathfrak{n}}(M) - 1$$

and

$$\mathrm{depth}_{\mathfrak{n}}(\overline{M}/y\overline{M}) = \mathrm{depth}_{\mathfrak{n}}(M/mM) - 1,$$

the result follows. $\square$

Lemma. *Let (R, m, K) to $(S, \mathfrak{n}, L)$ be a flat local map of local rings.*

(a) *$R(t) \to S(t)$ is flat, where t is an indeterminate.*

(b) *$\widehat{R} \to \widehat{S}$ is flat.*

Proof. For (a), $R \otimes_{\mathbb{Z}} \mathbb{Z}[t] \to S \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ is flat by base change, so that $R[t] \to S[t]$ is flat, and $S[t] \to S(t)$ is a localization, and so flat. Hence, $S(t)$ is flat over $R[t]$, and the map factors $R[t] \to R(t) \to S(t)$. Whenver B is flat over A and the map factors $A \to W^{-1}A \to T$, T is also flat over $W^{-1}A$. This follows form the fact that for $(W^{-1}A)$-modules $0 \to N \hookrightarrow M$, the map $T \otimes_{W^{-1}A} N \to T \otimes_{W^{-1}A} N$ may be identified with $T \otimes_A N \to T \otimes_A M$. To see this, note that we have a map $T \otimes_A M \to T \otimes_{W^{-1}A} M$, and the kernel is spanned by elements of the form $w^{-1}u \otimes v - u \otimes w^{-1}v$. But since w in invertible in T, we can prove that this is 0 by multiplying by w^2, which yields $wu \otimes v - u \otimes wv = 0$. This proves (a).

To prove (b), note that it suffices to prove that $0 \to N \hookrightarrow M$, a map of $\widehat{R}$-modules, remains injective after applying $\widehat{S} \otimes_{\widehat{R}} \_$ in the case where N and M are finitely generated. Given a counterexample, we can choose $u \in \widehat{S} \otimes_{\widehat{R}} N$ that is not 0 and is killed when mapped into $\widehat{S} \otimes_{\widehat{R}} M$. We can choose k so large that $u \notin m^k(\widehat{S} \otimes_{\widehat{R}} N)$, and, by the Artin-Rees lemma, we can choose n so large that $m^n M \cap N \subseteq m^k N$. Then there is a commutative diagram

$$\begin{array}{ccc} N & \hookrightarrow & M \\ \downarrow & & \downarrow \\ N/(m^n M \cap N) & \hookrightarrow & M/m^n M \end{array}$$

and we may apply $\widehat{S} \otimes_{\widehat{R}} \_$ to see that the image of u in $\widehat{S} \otimes_{\widehat{R}} \big(N/(m^n M \cap N)\big)$ is nonzero (even if we map further to $\widehat{S} \otimes_{\widehat{R}} (N/m^k M)$), but maps to 0 in $\widehat{S} \otimes_{\widehat{R}} (M/m^n M)$. When applied to maps of finite length $\widehat{R}$-modules, the functor $\widehat{S} \otimes_{\widehat{R}} \_$ preserves injectivity because $R \to S \to \widehat{S}$ is flat, and $\widehat{S} \otimes_{\widehat{R}} \_$ and $\widehat{S} \otimes_R \_$ are the same functor on finite length $\widehat{R}$-modules V: we have that $\widehat{R} \otimes_R V \cong V$ since V is killed by m^s for some s and $\widehat{R}/m^s\widehat{R} \cong R/m^s$, and so, by the associativity of tensor,

$$\widehat{S} \otimes_{\widehat{R}} V \cong \widehat{S} \otimes_{\widehat{R}} (\widehat{R} \otimes_R V) \cong \widehat{S} \otimes_R V. \qquad \square$$

We can now make several reductions in studying Lech's conjecture.

Theorem. *In order to prove Lech's conjecture that $e(R) \le e(S)$ when $(R, m, K) \to (S, \mathfrak{n}, L)$ is flat local and R has dimension d, it suffices to prove the case where* $\dim(S) = \dim(R) = d$*, R and S are both complete with infinite residue class field, S has algebraically closed residue class field, R is a domain, and S has pure dimension.*

Proof. By the Lemma above and the final Proposition in the Lecture Notes of October 20, we can replace R and S by $R(t)$ and $S(t)$ and so assume that the residue class fields are infinite. Likewise, we can replace R and S by their completions. We can choose a minimal prime Q of mS such that $\dim(S/Q) = \dim(S/mS)$. By the Lemma on p. 1 of the Lecture Notes of October 30, we have that $\mathrm{height}(Q) = \mathrm{height}(m) = \dim(R)$. Since $\dim(S) = \dim(S/mS) + \dim(R)$, we have that $\dim(S) = \dim(S/Q) + \mathrm{height}(Q)$. By the Theorem on behavior of multiplicities under localization in complete local rings, we then have $e(S_Q) \le e(S)$. Thus, if $e(R) \le e(S_Q)$ we have $e(R) \le e(S)$ as well. It follows that we may replace S by S_Q and so we may assume that $\dim(S) = \dim(R) = d$. We may have lost completeness, but we may complete again. By the second Proposition on p. 1 of the Lecture Notes of November 1, we can give a local flat map $(S, \mathfrak{n}, L) \to (S', \mathfrak{n}', L')$ such that S' is complete, $\mathfrak{n}' = \mathfrak{n}S'$ and L' is algebraically closed. Thus, we may assume that S has an algebraically closed residue class field.

We can give a filtration of R by prime cyclic modules R/P_i, $1 \le i \le h$. Then $e(R)$ is the sum of the $e(R/P_i)$ for those i such that $\dim(R/P_i) = \dim(R)$. Tensoring with S over R gives a corresponding filtration of S by modules S/P_iS, and $e(S)$ is the sum of the $e(S/P_iS)$ for those i such that $\dim(S/P_iS) = \dim(S)$. Since $\dim(S) = \dim(R) + \dim(S/mS)$ and, for each i, $\dim(S/P_iS) = \dim(R/P_i) + \dim(S/mS)$, the values of i such that $\dim(R/P_i) = \dim(R)$ are precisely those such that $\dim(S/P_iS) = \dim(S)$. Thus, it suffices to consider the case where R is a complete local domain.

If $\dim(R) = \dim(S)$ and R is a complete local domain, then it follows that S has pure dimension. We use induction on the dimension. If $\dim(R) = 0$ then $\dim(S) = 0$ and the result is clear. Let R' be the normalization of R. $R' \otimes_R S$ is faithfully flat over R' and still local (the maximal ideal of R' is nilpotent modulo the maximal ideal of R). Moreover, $S \subseteq R' \otimes_R S$, so that we may assume that R is normal. Suppose that S contains an S-submodule of dimension smaller than S, say J, and choose J maximum, so that S/J has pure dimension d. Then R does not meet J, and so injects into S/J. Choose $x \in m - \{0\}$. Then x is a nonzerodivisor in R, and, hence a nonzerodivisor on S and on J. It is also a nonzerodivisor on S/J, for any submodule killed by x would be a module over S/xS, and hence of smaller dimension. It follows that

$$0 \to xJ \to xS \to x(S/J) \to 0$$

is exact, and we get that

$$0 \to J/xJ \to S/xS \to (S/J)/x(S/J) \to 0$$

is exact. Then

$$\dim(J/xJ) \leq \dim(J) - 1 < \dim(S) - 1 = \dim(S/xS).$$

Therefore, S/xS does not have pure dimension. Because all associated primes of xR have height one, R/xR has a filtration whose factors are torsion-free modules over rings R/P_i of dimension $\dim(R)-1$, where the P_i are the minimal primes of x. By the induction hypothesis, every S/P_iS has pure dimension. Since a finitely generated torsion-free module over R/P_i embeds in a finitely generated free module over R/P_i, the tensor product of a finitely generated torsion-free module over R/P_i with S also has pure dimension. Thus, S/xS has a filtration whose factors are modules of pure dimension, and so has pure dimension itself. This contradiction establishes the result. $\square$

One approach to obtaining a class of local rings R for which Lech's conjecture holds for every flat local map $R \to S$ is via the notion of a *linear maximal Cohen-Macaulay module.* Recall that over a local ring (R, m, K), a module M is a Cohen-Macaulay module if it is finitely generated, nonzero, and $\text{depth}_m M = \dim(M)$. In particular, M is Cohen-Macaulay module over R if and only if it is Cohen-Macaulay module over R/I, where $I = \text{Ann}_R M$. E.g., the residue class field $K = R/m$ is always Cohen-Macaulay module over R. By a maximal Cohen-Macaulay module we mean a Cohen-Macaulay module module whose dimension is equal to $\dim(R)$. It is not known whether every excellent local ring has a maximal Cohen-Macaulay module: this is an open question in dimension 3 in all characteristics.

We write $\nu(M)$ for the least number of generators of the R-module M. If M is finitely generated over a local (or quasi-local) ring (R, m, K), Nakayama's lemma implies that $\nu(M) = \dim_K(M/mM)$.

Note the following fact, which has proved useful in studying Lech's conjecture:

Proposition. *Let (R, m, K) be local and let M be a maximal Cohen-Macaulay module. Then $e(M) \geq \nu(M)$.*

Proof. We may replace R by $R(t)$ and M by $R(t) \otimes_R M$ if necessary and so assume that the residue class field of R is infinite. Let $I = (x_1, \ldots, x_d)$ be a minimal reduction of m, where $d = \dim(R)$. Then $e(M) = \ell(M/IM) \geq \ell(M/mM) = \nu(M)$. $\square$

We shall call M a *linear maximal Cohen-Macaulay module* over the local ring $(R,\, m,\, K)$ if it is a maximal Cohen-Macaulay module and $e(M) = \nu(M)$. Because of the inequality in the Proposition just above, the term *maximally generated* maximal Cohen-Macaulay module is also used in the literature, as well as *top-heavy* maximal Cohen-Macaulay module and *Ulrich* maximal Cohen-Macaulay module. The idea of the proof of the Proposition above also yields:

Proposition. *Suppose that M is a maximal Cohen-Macaulay module over a local ring $(R,\, m,\, K)$ and that K is infinite or, at least, the m has a minimal reduction I generated by a system of parameters $x_1, \, \ldots\, , x_d$. Then:*

(a) *M is a linear maximal Cohen-Macaulay module if and only if $mM = IM$.*

(b) *If M is a linear maximal Cohen-Macaulay module then $m^n M = I^n M$ for all $n \in \mathbb{N}$.*

(c) *If M is a linear maximal Cohen-Macaulay module then $\mathrm{gr}_m(M)$ is a Cohen-Macaulay module over $\mathrm{gr}_m(R)$.*

Proof. (a) Since $IM \subseteq mM \subseteq M$ we have that

$$e(M) = \ell(M/IM) = \ell(M/mM) + \ell(mM/IM) = \nu(M) + \ell(mM/IM).$$

Hence, $e(M) = \nu(M)$ if and only if $\ell(mM/IM) = 0$, i.e., if and only if $mM = IM$.

Part (b) follows by induction on n: if $m^n M = I^n M$ then

$$m^{n+1}M = m^n mM = m^n(IM) = I(m^n M) = I(I^n M) = I^{n+1}M.$$

For part (c), observe that $\mathrm{gr}_m M = \mathrm{gr}_I M$, by part (b), and the result is then immediate from part (d) of the Proposition on p. 2 of the Lecture Notes of October 23, which identifies $\mathrm{gr}_I(M)$ with

$$(M/IM) \otimes_{R/I} (R/I)[X_1, \, \ldots\, , X_d],$$

where X_i is the image of x_i in I/I^2, $1 \leq i \leq d$, and $X_1, \, \ldots\, , X_d$ are algebraically independent over R/I. $\square$