Math 711: Lecture of November 10, 2006

Before proceeding further with our study of Lech's conjecture, we want to discuss a result known as the *associativity* of multiplicities whose proof uses Lech's theorem on computing multiplicities using ideals generated by powers of parameters.

Before we prove this result, we want to make some remarks on the behavior of limits of real-valued functions of two integer variables: these might as well be positive, since we will be taking limits as the variables approach $+\infty$ either jointly or independently. Let $G(m, n)$ be such a function, and suppose that

$$\lim_{(m,n)\to\infty} G(m, n)$$

exists, and also that the iterated limit

$$\lim_{m\to\infty}\bigl(\lim_{n\to\infty} G(m, n)\bigr)$$

exists as well. Then they are equal. If the joint limit is L, then, given $\epsilon > 0$ there exists N such that for all $m, n \geq N$, $|L - G(m, n)| < \epsilon$. It follows that each for $m \geq n$, $L_m = \lim\limits_{n\to\infty} G(m, n)$, which we are assuming exists, is such that $|L - L_m| \leq \epsilon$, since all of the values of $G(m, n)$ are at distance at most ϵ from L for all $m \geq N$. It also follows that the iterated limit must have a value that is at distance at most ϵ from L. Since this is true for all $\epsilon > 0$, the iterated limit must be L.

Note however, that when the joint limit exists, it is possible that neither iterated limit exists. E.g., for $m, n \geq 1$ we can let

$$G(m,n) = 1/\min\{m, n\}$$

if $m + n$ is even and 0 if $m + n$ is odd. For any fixed m, the function takes on the values $1/m$ and 0 alternately for large n, and the iterated limit does not exist.

On the other hand, observe that we can define $G(m,n) = 1$ if $m \geq n$ and $G(m,n) = 0$ if $m < n$. The iterated limits both exist, but are different, and the joint limit does not exist. If, alternatively, we define $G(m, n) = 0$ when $m = n$ and $G(m, n) = 1$ when $m \neq n$, both iterated limits exist and are equal, but the joint limit does not.

Note that the Corollary on p. 5 of the Lecture Notes of October 20 is a special case of the following Theorem, namely the case where $r = 0$. It is also used in the proof.

Theorem (associativity of multiplicities). *Let M be a module of dimenson d over a local ring R of dimension d. Let $x_1, \ldots, x_d$ be a system of parameters for R, and let $I = (x_1, \ldots, x_d)R$. Let r, s be nonnegative integers such that $r+s = d$. Let $\mathfrak{A} = (x_1, \ldots, x_r)R$*

and $\mathfrak{B} = (x_{r+1}, \ldots, x_d)R$. *Let* $\mathcal{P}$ *be the set of minimal primes of* $x_1, \ldots, x_r$ *such that* $\dim M_P + \dim(R/P) = d$. *Then*

$$e_I(M) = \sum_{P \in \mathcal{P}} e_{\mathfrak{B}}(R/P)e_{\mathfrak{A}}(M_P).$$

Proof. Let $\mathfrak{A}_m = (x_1^m, \ldots, x_r^m)R$ and $\mathfrak{B}_n = (x_{r+1}^n, \ldots, x_d^n)R$. Consequently, by Lech's result on calculation of multiplicities, which is the Theorem on p. 3 of the Lecture Notes of October 25,

$$e_I(M) = \lim_{m,n\to\infty} \frac{\ell\big(M/(\mathfrak{A}_m + \mathfrak{B}_n)M\big)}{m^r n^s}.$$

In this instance we know that the joint limit exists. The iterated limit exists as well, since we have

$$\lim_{m\to\infty}\big(\frac{1}{m^r}\lim_{n\to\infty}\frac{\ell\big(M/(\mathfrak{A}_m + \mathfrak{B}_n)M\big)}{n^s}\big) = \lim_{m\to\infty}\frac{e_{\mathfrak{B}}(M/\mathfrak{A}_m M)}{m^r}.$$

Since M and R have dimension d and $x_1, \ldots, x_d$ is a system of parameters, so is $x_1^m, \ldots, x_r^m, x_{r+1}^n, \ldots, x_d^n$, and $M/\mathfrak{A}_m M$ will have dimension s. P will be a (necessarily minimal) prime of the support of $M/\mathfrak{A}_m M$ such that $\dim(R/P) = \dim(M/\mathfrak{A}M)$ if and only if P contains $\mathfrak{A}$, P contains $\mathrm{Supp}(M)$, and $\dim(R/P) = s$. Since $\dim(R/\mathfrak{A}) = s$, P must be a minimal prime of $\mathfrak{A}$. Let $\mathcal{Q}$ denote the set of minimal primes of $\mathfrak{A}$ such that $\dim(R/P) = s$. By the Corollary on p. 5 of the Lecture Notes of October 20,

$$e_{\mathfrak{B}}(M/\mathfrak{A}_m M) = \sum_{P\in\mathcal{Q}} e_{\mathfrak{B}}(R/P)\ell_{R_P}(M_P/\mathfrak{A}_m M_P).$$

Note that since P is a minimal prime of $\mathfrak{A}$, $\mathfrak{A}$ expands to an ideal primary to the maximal ideal in R_P. Consequently,

$$\lim_{m\to\infty}\frac{e_{\mathfrak{B}}(M/\mathfrak{A}_m M)}{m^r} = \sum_{P\in\mathcal{Q}} e_{\mathfrak{B}}(R/P)\lim_{m\to\infty}\frac{\ell_{R_P}(M_P/\mathfrak{A}_m M_P)}{m^r}.$$

We now see that the iterated limit exists. Note that

$$\lim_{m\to\infty}\frac{\ell_{R_P}(M_P/\mathfrak{A}_m M_P)}{m^r}$$

is 0 if $\dim(M_P) < r$, and $e_{\mathfrak{A}}(M_P)$ otherwise. Therefore we need only sum over those minimal primes P of $\mathfrak{A}$ in $\mathrm{Supp}(M)$ such that $\dim(M_P) = r$ and $\dim(R/P) = s$. The latter two conditions are equivalent to the condition $\dim(M_P) + \dim(R/P) = d$ given that $P \in \mathrm{Supp}(M)$ contains $\mathfrak{A}$, since $\dim(M_P) \le \dim(R_P) \le r$ and $\dim(R/P) \le \dim(M/\mathfrak{A}M) = s$.

However, if P is a minimal prime of $\mathfrak{A}$, we do not need to require that $P \in \mathrm{Supp}\,(M)$, for if not $M_P = 0$ and the term corresponding to P does not affect the sum. Therefore, the value does not change if we sum over primes $P \in \mathcal{P}$, and this yields

$$e_I(M) = \sum_{P \in \mathcal{P}} e_{\mathfrak{B}}(R/P)e_{\mathfrak{A}}(M_P),$$

as required. $\square$

Our next objective is to obtain some classes of local rings R such that Lech's conjecture always holds for flat local maps $R \to S$. Lech proved the result for the case where $\dim\,(R) \le 2$. However, we shall first focus on cases that are given by the existence of linear maximal Cohen-Macaulay modules and related ideas.

We shall say that an R-module M has *rank* ρ if there is a short exact sequence

$$0 \to R^\rho \to M \to C \to 0$$

such that $\dim\,(C) < \dim\,(R)$. If M has rank ρ for some ρ, we shall say that M is an R-module *with rank*. When R is a domain, M always has rank equal to the maximum number of elements of M linearly independent over R. When R has a module M of rank ρ, $e(M) = \rho e(R)$, by the additivity of multiplicities.

The following result is very easy, but very important from our point of view.

Theorem. *Let $(R,\,m,\,K)$ be a local ring that has a linear maximal Cohen-Macaulay module of rank ρ. Then for every flat local map $(R,\,m,\,K) \to (S,\,\mathfrak{n},\,L)$, $e(R) \le e(S)$.*

Proof. As observed earlier, we may assume that mR is $\mathfrak{n}$-primary. It follows that $S \otimes M$ is Cohen-Macaulay over S, and also has rank ρ. Then

$$e(R) = e(M) = \frac{1}{\rho}\,\nu(M) = \frac{1}{\rho}\,\nu(S \otimes_R M) \le \frac{1}{\rho}\,e(S \otimes M) = e(S). \qquad \square$$

Therefore, Lech's conjecture holds for any ring R that admits a maximal Cohen-Macaulay module M such that $e(M) = \nu(M)$. Consequently, we shall focus for a while on the problem of constructing linear maximal Cohen-Macaulay modules.

However, we first want to point out the following idea, which has also been used to settle Lech's conjecture in important cases. Suppose that R and S are as in the statement of the Theorem just above, and that M is any Cohen-Macaulay module with rank, say ρ. Then

$$e(S) = \frac{1}{\rho}\,e(S \otimes_R M) \ge \frac{1}{\rho}\,\nu(S \otimes_R M) = \frac{1}{\rho}\,\nu(M) = \frac{1}{\rho}\,e(M)\,\frac{\nu(M)}{e(M)} = \frac{\nu(M)}{e(M)}\,e(R).$$

Hence:

Theorem. *Let $(R,\, m,\, K)$ be a local ring that has a sequence of Cohen-Macaulay modules $\{M_n\}$ with rank such that*

$$\lim_{n\to\infty} \frac{\nu(M)}{e(M)} = 1.$$

Then for every flat local map $(R,\, m,\, K) \to (S,\, \mathfrak{n},\, L)$, $e(R) \le e(S)$. $\square$

Remark. In fact, it suffices if there is any Cohen-Macaulay module M with rank such that $\dfrac{\nu(M)}{e(M)} > \dfrac{e(R)-1}{e(R)}$.

Lech's conjecture is easy in dimension 0: it reduces to the case where R is a local domain, and therefore a field. Thus, $e(R) = 1 \le e(S)$.

In dimension one, we need only consider the case of a local domain.

Theorem. *Let $(R,\, m,\, K)$ be a local domain with an infinite residue class field. Then m^k is a linear maximal Cohen-Macaulay module for all sufficiently large $k \gg 0$*

Therefore Lech's conjecture holds for all local rings R of dimension at most one.

Proof. Since the Hilbert polynomial $\ell(R/m^{n+1}$ is $e(R)n + c$ for some constant c, it follows that $\ell(m^k/m^{k+1}) = e(R)$ for all $k \gg 0$. But this is also $\nu(m^k)$. Since R is a one-dimensional domain, any nonzero torsion-free module is Cohen-Macaulay. $\square$

Remark. When the residue field is infinite, we can argue alternatively that m has a reduction xR, and then for sufficiently large k, $m^{k+1} = xm^k$.

We shall eventually prove that if $(R,\, m,\, K)$ is local such that R has a linear maximal Cohen-Macaulay module, and $f \in m$ is such that the leading form of f in $\mathrm{gr}_m(R)$ is a nonzerodivisor, then R/fR has a linear maximal Cohen-Macaulay module. It will follow that if R is a *strict complete intersection*, i.e., has the form $T/(f_1,\, \dots\, , f_h)$ where $(T, \mathcal{M})$ is regular and the leading forms of the f_j form a regular sequence in $\mathrm{gr}_{\mathcal{M}}T$, then R has a linear maximal Cohen-Macaulay module. This will take a considerable effort.