Math 711: Lecture of November 13, 2006

We are going to use certain matrix factorizations to construct linear maximal Cohen-Macaulay modules over hypersurfaces.

Discussion: Cohen-Macaulay modules over hypersurfaces R/fR of finite projective dimension over R. In order to illustrate the connection, we first consider the following relatively simple situation. Let (R, m, K) be a Cohen-Macaulay local ring of dimension n, let $f \in m$ be a nonzerodivisor, and suppose that we want to study maximal Cohen-Macaulay modules over R/fR that have finite projective dimension over R. Such a module M has depth $n-1$, and so $\mathrm{pd}_R M = 1$. Thus, there is a minimal free resolution

$$0 \to R^h \to R^s \to M \to 0$$

of M over R. If we localize at f, since $M_f = 0$, we see that $h = s$, and so M is given as the cokernel of an $s \times s$ matrix α. Let e_i be the i th standard basis vector for R^s, written as a column. Then for every i, $1 \le i \le s$, we have, since $fM = 0$, that fe_i is in the column space of α, which means that there is a column vector B_i such that $\alpha B_i = fe_i$. If we form the $s \times s$ matrix β whose columns are $B_1, \ldots, B_n$, we have that $\alpha\beta = f\boldsymbol{I}_s$, where $\boldsymbol{I}_s$ is the $s \times s$ identity matrix. Over the ring $R_f \supseteq R$, $\beta = f\alpha^{-1}$, which implies that α and β commute, i.e., $\alpha\beta = \beta\alpha = f\boldsymbol{I}_s$.

Let $\overline{}$ indicate images in R/fR. We claim that the complex

$$\cdots \xrightarrow{\overline{\beta}} \overline{R}^s \xrightarrow{\overline{\alpha}} \overline{R}^s \xrightarrow{\overline{\beta}} \overline{R}^s \xrightarrow{\overline{\alpha}} \overline{R}^s \to 0,$$

whose augmentation is M, is acyclic. Suppose $v \in R^s$ represents a vector in the kernel of α (the argument for β is identical). Then $\alpha(v)$ vanishes mod fR^s, and so $\alpha(v) = fu$. Then $\beta\alpha(v) = \beta(fu) = f\beta(u)$, but $\beta\alpha = f\boldsymbol{I}_s$, and so we have $fv = f\beta(u)$. Since f is not a zerodivisor, $v = \beta(u)$, and exactness follows.

Given M, we obtain a matrix factorization. Conversely, given a matrix factorization of $f\boldsymbol{I}_s$, the argument we just gave shows that the complex

$$\cdots \xrightarrow{\overline{\beta}} \overline{R}^s \xrightarrow{\overline{\alpha}} \overline{R}^s \xrightarrow{\overline{\beta}} \overline{R}^s \xrightarrow{\overline{\alpha}} \overline{R}^s \to 0,$$

is exact over $\overline{R} = R/fR$. Call the augmentation M. Then $M = \mathrm{Coker}\,(\alpha)$ is its own k th module of syzygies for arbitarily large k. This implies that $\mathrm{depth}_m(M) = n-1$: over a Cohen-Macaulay ring, if a module has depth b smaller than that of the ring, its first module of syzygies has depth $b+1$. Once the module has depth equal to that of the ring, the modules of syzygies, if nonzero, continue to have depth equal to that of the ring (in the case of R/fR, the eventual depth is $n-1$). We have therefore established a correspondence between matrix factorizations of f into two factors and Cohen-Macaulay modules over R/fR that have finite projective dimension over R.

In our eventual construction of linear maximal Cohen-Macaulay modules over hypersurfaces, we shall make use of matrix factorizations with large numbers of factors.

By a *matrix factorization* of $f \in R$ over R of *size* s with d *factors* we mean a d-tuple of matrices $\alpha = (\alpha_1, \ldots, \alpha_d)$ over R such that

$$f\boldsymbol{I}_s = \alpha_1 \cdots \alpha_d$$

and satisfying the additional condition that for all i,

$$f\boldsymbol{I}_s = \alpha_i \alpha_{i+1} \cdots \alpha_d \alpha_1 \alpha_2 \cdots \alpha_{i-1}$$

as well. Here, it will be convenient to interpret the subscripts mod d. Note that the weak commutativity condition is automatic if f is a nonzerodivisor in R, for then

$$(\alpha_1 \cdots \alpha_{i-1})^{-1} = f\alpha_i \alpha_{i+1} \cdots \alpha_d$$

for all i, which implies that $\alpha_1 \cdots \alpha_{i-1}$ and $\alpha_i \alpha_{i+1} \cdots \alpha_d$ commute for all i. If α is either a matrix or a d-tuple of matrices, $I(\alpha) = I_1(\alpha)$ denotes the ideal generated by the entries of α or by the entries of all of the matrices occurring in α.

We define two matrix factorizations of size s with d factors, say $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$, to be *equivalent* if there are invertible $s \times s$ matrices $\gamma_0, \gamma_1, \ldots, \gamma_d$ over R such that $\gamma_0 = \gamma_d$ and for all i, $1 \leq i \leq d$, we have $\beta_i = \gamma_{i-1}\alpha_i\gamma_i^{-1}$.

Our goal is to prove that if $I \subseteq R$ and $f \in I^d$ for $d \geq 2$ then f has a matrix factorization $\alpha = (\alpha_1, \ldots, \alpha_d)$ of some size s with d factors such that $I(\alpha) = I$. Moreover, we shall see that by increasing s this can even be done in such a way that all of the matrices α_i are the same, and $I(\alpha_i) = I$ for all i.

The argument will involve solving the problem for a "generic form" of degree s in indeterminates over $\mathbb{Z}$, and then specializing.

We shall use the theory of Clifford algebras and Clifford modules over a field K to prove the existence of matrix factorizations.

Discussion: the definition of Clifford algebras. Let K be a field, let L be the vector space spanned by n variables $X_1, \ldots, X_n$, and let $V = L^* = \mathrm{Hom}_K(V, K)$ be the space of linear functionals on L with dual basis $e_1, \ldots, e_n$. Let f be a form of degree d in $X_1, \ldots, X_d$ such that f does not vanish identically on K^d. This is automatic when K is infinite. Our main interest is in the case where $K = \mathbb{Q}$.

The *Clifford algebra* over K corresponding to f, which we denote $C(f)$, is defined as follows. Let $\mathcal{T}(V)$ denote the tensor algebra of V over K, which is an $\mathbb{N}$-graded associative algebra over K such that $[\mathcal{T}(V)]_h$ is the h-fold tensor product $V \otimes_K V \otimes_K \cdots \otimes_K V$, where there are h copies of V. We may alternatively write $[\mathcal{T}(V)]_h = V^{\otimes h}$. Note that $\mathcal{T}(V)]_0 = K$, and $[\mathcal{T}(V)]_1 = V$, which generates $\mathcal{T}(V)$ over K. The multiplication is such that

$$(v_1 \otimes \cdots \otimes v_a)(w_1 \otimes \cdots \otimes w_b) = (v_1 \otimes \cdots \otimes v_a \otimes w_1 \otimes \cdots \otimes w_b).$$

$C(f)$ is defined as the quotient of $\mathcal{T}(V)$ by the two-sided ideal generated by all elements of the form

$$(c_1e_1 + \cdots + c_ne_n)^d - f(c_1, \dots, c_n)$$

for all $(c_1, \dots, c_n) \in K^n$. Thus, $\mathcal{T}(f)$ is universal with respect to the property the dth power of every linear form is a scalar whose value is determined by applying f to the coefficients of the form.

The most common use of these algebras is in the case where $d = 2$, but it will be very important to consider larger values of d here.

We note that the the Clifford algebra $C(f)$ is $\mathbb{Z}_d$-graded: this follows from the fact that every homogeneous component of every generator has degree 0 or d. When C is a $\mathbb{Z}_d$-graded algebra or module, and i is an integer, we may write either $[C]_i$ or $[C]_{[i]}$ for the component of C in degree $[i]$, where $[i]$ is the class of i modulo d.

By a *Clifford module* over a Clifford algebra $C(f)$ we mean a $\mathbb{Z}_d$-graded $C(f)$-module M such that M is a nonzero finite-dimensional K-vector space. It is not at all clear whether a Clifford algebra has a Clifford module. The following result makes a strong connection with matrix factorization. We may write either $[M]_i$ or M_i for the graded component in degree i.

Proposition. *Let K be an field, let L be the vector space spanned over K by n indeterminates $X_1, \dots, X_n$, let f be a form of degree $d \geq 2$ that does not vanish identically on K^n, which is always the case when f is nonzero and K is infinite, let $V = L^*$, and let $C(f)$ be the Clifford algebra of f over K.*

(a) *If M is a Clifford module, then all the M_i have the sameK-vector space dimension, say s.*

(b) *If K is infinite, then there is a bijective correspondence between isomorphism classes of Clifford modules M over $C(f)$ such that every M_i has dimension $s \geq 1$ over K and equivalence classes of of matrix factorizations of f over K.*

Proof. For part (a), suppose $F(c_1, \dots, c_n) \neq 0$ and let $v = c_1e_1 + \cdots + c_ne_n \in V$. Multiplication by v gives a K-linear map $M_i \to M_{i+1}$ for every i. The composition of all of these maps is multiplication by v^n, which is the same as multiplication by $f(c_1, \dots, c_n) = a \in K - \{0\}$. Since v^n is an automorphism of M, each map $M_i \to M_{i+1}$ given by multiplication by v is a K-linear bijection.

We now prove (b). Fix a K-basis for every M_i. Let η_{ij} be the matrix of multiplication by $e_j \in V$ as a map $M_i \to M_{i+1}$ with respect to the fixed bases. Let $\underline{c} = c_1, \dots, c_n$ be any elements of K. Then the matrix of multiplication by $v = c_1e_1 + \cdots c_ne_n$ mapping $M_i \to M_{i+1}$ is $\theta_i(\underline{c}) = c_1\eta_{i1} + \cdots + c_n\eta_{in}$. For all choices of $\underline{c}$, the composition of maps

$$\big(M_{i+d-1} \xrightarrow{\theta_{i+d-1}(\underline{c})} M_{i+d}\big) \circ \cdots \circ \big(M_{i+1} \xrightarrow{\theta_{i+1}(\underline{c})} M_{i+2}\big) \circ \big(M_i \xrightarrow{\theta_i(\underline{c})} M_{i+1}\big)$$

is multiplication by $f(c_1, \dots, c_n)$, i.e.,

$$(*) \qquad f\big((\underline{c})\big)\boldsymbol{I}_s = \theta_i(\underline{c})\theta_{i+1}(\underline{c}) \cdots \theta_{i+d-1}(\underline{c}).$$

Keep in mind here that $M_{i+d} = M_i$. Since K is infinite, it follows that $(*)$ holds when we replace the elements c_i by indeterminates. Thus, if for every i we let

$$\alpha_i' = X_1\eta_{i1} + \cdots + X_n\eta_{in},$$

which is a matrix of linear forms in the X_j over K, then we must have

$$(**) \qquad f\big(X_1, \ldots, X_n\big)\boldsymbol{I}_s = \alpha_{i-1}'(X_1, \ldots, X_n)\alpha_{i-2}'(X_1, \ldots, X_n)\cdots\alpha_i'(X_1, \ldots, X_n)$$

for all i. Thus gives a matrx factorization except that the matrices are numbered in reverse. If we let $\alpha_i = \alpha_{d+1-i}'$ we have a matrix factorization

Conversely, given a matrix factorization of size s, we may construct a Clifford module M with all components M_i isomorphic to K^s by letting the multiplication by the linear form

$$v = c_1e_1 + \cdots + c_ne_n$$

from the ith component to the $i+1$st component be the map whose matrix is obtained by the substitution $X_1 = c_1, \ldots, X_n = c_n$ in the matrix α_{d+1-i}.

The statement about isomorphism now follows from the fact that if $\gamma_1, \ldots, \gamma_d$ are change of basis matrices for the various $M_{d+1-i} = K^s$, the matrices α_i change to $\gamma_{i-1}\alpha_i\gamma_i^{-1}$. $\square$

Let C and C' be two $\mathbb{Z}_d$-graded associative K-algebras (this statement includes the hypothesis that K is in the center of each). Let $\Psi_d(t) \in \mathbb{Z}[t]$ denote the dth cyclotomic polynomial (we discuss these further later) over $\mathbb{Q}$, which is the minimal polynomial over $\mathbb{Q}$ of a primitive dth root of unity in $\mathbb{C}$. Assume that K contains a root ξ of $\Psi_d(t)$, which will, of course, be a dth root of unity, since $\Psi(t)$ divides $t^d - 1$ even in $\mathbb{Z}[t]$. K can always be enlarged by a finite algebraic extension to contain such an element ξ. If K has characteristic 0, ξ is simply a primitive dth root of unity.

We then define the *twisted* tensor product $C \otimes_K C'$ to be the $\mathbb{Z}_d$-graded K-algebra which, as a K-vector space, is simply $C \otimes_K C'$, graded so that

$$[C \otimes_K C']_i = \bigoplus_{j+k=i} [C]_i \otimes_K [C']_j$$

where i, j, and k are in $\mathbb{Z}_d$, and with multiplication such that, if $u, v \in C$ are forms and $u', v' \in C'$ are forms, then

$$(\#) \qquad (u \otimes u')(v \otimes v') = \xi^{\deg(u')\deg(v)}(uv) \otimes (u'v').$$

It is easy to check that multiplication is associative for triples of elements each of which is a tensor product of two forms, and the general case follows readily. Notice in particular that if u and u' are 1-forms, then

$$(1 \otimes u')(u \otimes 1) = \xi(u \otimes 1)(1 \otimes u').$$

Quite similarly, we can give essentially the same definition for the *twisted tensor product* $M \otimes_K M'$ of a $\mathbb{Z}_d$-graded C-module M and a $\mathbb{Z}_d$-graded C'-module M', which will be a $\mathbb{Z}_d$-graded module over $C \otimes_K C'$. One has to give the action of $C \otimes_K C'$ on $M \otimes_K M'$. The formula is the same as given in $(\#)$, but now $u \in C$, $u \in C'$, $v \in M$ and $v' \in M'$ are forms.