Math 711: Lecture of November 20, 2006

The following result can be deduced easily from the Buchsbaum-Eisenbud acyclicity criterion, but we give a short, self-contained argument.

Proposition. *Let (R, m, K) be a local ring and N a maximal Cohen-Macaulay module over R. If M is a finitely generated R-module of finite projective dimension over R, then $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

Proof. For any prime ideal P in $\mathrm{Supp}\,(N)$, N_P is again a maximal Cohen-Macaulay module over R_P. (This is clear if $\mathrm{height}\,(P) = 0$. If $\mathrm{height}\,(P) > 0$, choose $x \in P$ not in any minimal prime of R. Then x is not a zerodivisor on N, and the result follows by Noetherian induction, since N/xN will be a maximal Cohen-Macaulay module for R/xR.) Let $\mathrm{pd}_R(M) = h$. Then $\mathrm{Tor}_i^R(M,\, N) = 0$ for $i > h$. If we have a a counterexample, we can localize at a minimal prime P of $\bigoplus_{i=1}^h \mathrm{Tor}_i^R(M,\, N)$. Thus, we may assume without loss of generality that all of the non-vanishing $\mathrm{Tor}_i^R(M,\, N)$ for $i \geq 1$ have finite length, and we can choose i as large as possible for which one of these Tor modules is not 0. If

$$0 \to G_h \to \cdots \to G_0 \to 0$$

is a minimal free resolution of M over R, the modules $\mathrm{Tor}_i^R(M,\, N)$ are the homology modules of the complex

$$0 \to G_h \otimes_R N \to \cdots \to G_0 \otimes_R N \to 0.$$

Let d_j denote the map

$$G_j \otimes_R N \to G_{j-1} \otimes_R N.$$

Let $Z_j = \mathrm{Ker}\,(d_j)$ and let $B_j = \mathrm{Im}\,(d_{j+1})$. Thus, $Z_j = B_j$ for $j > i$. Note that we cannot have $\dim(R) = 0$, for then $\mathrm{pd}_R M \leq \mathrm{depth}_m(R) = 0$, and M is free. Thus, we may assume $\dim(R) \geq 1$. Also note that we cannot have $i = h$, because $\mathrm{Tor}_R^h(M,\, N) \subseteq G_h \otimes_R N$, a finite direct sum of copies of N, and has no submodule of finite length, since $\mathrm{depth}_m N = \dim(R) > 0$. Then we have a short exact sequence

$$0 \to B_i \to Z_i \to \mathrm{Tor}_i^R(M,\, N) \to 0$$

and an exact sequence

$$0 \to G_h \otimes_R N \to \cdots \to G_{i+1} \otimes_R N \to B_i \to 0.$$

Since $\mathrm{Tor}_i^R(M,\, N)$ is a nonzero module of depth 0, $Z_i \neq 0$, and $Z_i \subseteq G_i \otimes N$ has depth at least one. It follows that $\mathrm{depth}_m(B_i) = 1$. The exact sequences

$$0 \to B_j \to G_j \otimes_R N \to B_{j-i} \to 0$$

for $j > i$ enable us to see successively that $\mathrm{depth}_m B_{i+1} = 2$, $\mathrm{depth}_m B_{i+2} = 3$ and, eventually, $\mathrm{depth}_m B_{h-1} = (h-1) - (i-1) = h - i$. But $B_{h-1} = G_h \otimes_R N$ has depth $\dim(R) \geq \mathrm{depth}_m R \geq h > h - i$, since $i \geq 1$, a contradiction. $\square$

We next observe the following result related to the final Corollary of the Lecture of November 17.

Lemma. *Let (R, m, K) be local, and $f \in m^d - m^{d+1}$ be such that $\mathcal{L}(f)$ is part of a homogeneous system of parameters for $\mathrm{gr}_m(R)$. Let N be a linear maximal Cohen-Macaulay module. Then $e(N/fN) = de(N)$.*

Proof. Without loss of generality we may replace R by $R(t)$ and N by $R(t) \otimes_R N$, and so assume that we have an infinite residue class field. Let $\dim(R) = r$, and let $x_1, \ldots, x_r$ be a minimal reduction of m. Then $\mathrm{gr}_N \cong (N/mN) \otimes_K K[X_1, \ldots, X_r]$ is Cohen-Macaulay of depth r over $\mathrm{gr}_m(R)$: the images of the x_j in m/m^2 form a regular sequence. It follows that $\mathcal{L}(f)$, which has degree d, is a nonzerodivisor on $\mathrm{gr}_m(N)$, and we may apply part (b) of the final Corollary of the Lecture of November 17. $\square$

We are now ready to prove the result we are aiming for (cf. [J. Herzog, B. Ulrich, and J. Backelin, *Linear maximal Cohen-Macaulay modules over strict complete intersections*, Journal of Pure and Applied Algebra **71** (1991) 187–202.]

Theorem (Herzog, Ulrich, and Backelin). *Let (R, m, K) be a Cohen-Macaulay local ring that has a linear maximal Cohen-Macaulay module N. Let $f \in m^d - m^{d+1}$ be a nonzerodivisor such that its leading form $\mathcal{L}(f) \in \mathrm{gr}_m(R)$ is part of a homogeneous system of parameters for $\mathrm{gr}_m(R)$. Then R/fR has a linear maximal Cohen-Macaulay module.*

Proof. We adopt the notation of the Theorem on p. 3 of the Lecture of November 17, so that we have a chain

$$R^s = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_d = fR^s$$

as in the statement of that Theorem. Thus, the modules

$$0 \subseteq G_{d-1}/G_d \subseteq \cdots \subseteq G_i/G_d \subseteq \cdots \subseteq G_0/G_d$$

give an ascending filtration of $G_0/G_d \cong R^s/fR^s$ in which every factor module M_i is a maximal Cohen-Macaulay module over $\overline{R} = R/fR$ that is minimally generated by s elements and such that all of the modules G_i/G_d, G_0/G_i, and $M_i = G_i/G_{i+1}$ have finite projective dimension over R.

We shall show that the modules $G_i/G_d \otimes N$ give an ascending filtration of N^s/fN^s such that each factor $M_i \otimes_R N$ is a maximal Cohen-Macaulay module over $\overline{R}$. We then prove that at least one of $M_i \otimes_R N$ is a linear maximal Cohen-Macaulay module over $\overline{R}$.

First, we have short exact sequences

$$0 \to G_i/G_d \to G_0/G_d \to G_0/G_i \to 0,$$

$0 \leq i \leq d-1$ (this is the range for i throughout). These remain exact when we apply $\_ \otimes_R N$, since G_0/G_i has finite projective dimension over R and N is a maximal Cohen-Macaulay module: $\mathrm{Tor}_1(G_0/G_i, N) = 0$. This shows that each $(G_i/G_d) \otimes_R N$ embeds in

$$G_0/G_d \otimes_R N \cong \overline{R}^s \otimes_R N \cong (N/fN)^{\oplus s}.$$

The short exact sequences

$$0 \to G_{i+1}/G_d \to G_i/G_d \to M_i \to 0$$

likewise remain exact when we apply $\_ \otimes_R N$, and so it follows that the factors are the modules $M_i \otimes_R N$.

We want to prove that each $M_i \otimes_R N$ is a maximal Cohen-Macaulay module over $\overline{R}$. We use the notation of the proof of the Theorem on p. 3 of the Lecture Notes of November 17. Recall that the complex

$$\cdots \xrightarrow{\overline{\psi}_i'} \overline{R}^s \xrightarrow{\overline{\psi}_i} \overline{R}^s \xrightarrow{\overline{\psi}_i'} \overline{R}^s \xrightarrow{\overline{\psi}_i} \overline{R}^s \to 0$$

is acyclic, and that $\operatorname{Coker}\psi_i = R^s/G_i$, which is also $\operatorname{Coker}\overline{\psi_i}$. Moreover, we have that $\operatorname{Im}\overline{\psi_i} \cong G_i/G_d$. Therefore we have short exact sequences:

$$0 \to R^s/G_i \to \overline{R}^s \to G_i/G_d \to 0 \qquad \text{and} \qquad 0 \to G_i/G_d \to \overline{R}^s \to R^s/G_i \to 0$$

for all i, $0 \le i \le d-1$. Since these modules have finite projective dimension over R, both sequences remain exact when we apply $\_ \otimes_R N$, yielding

$$(*) \quad 0 \to (R^s/G_i) \otimes_R N \to \overline{N}^s \to (G_i/G_d) \otimes_R N \to 0$$

and

$$(**) \quad 0 \to (G_i/G_d) \otimes_R N \to \overline{N}^s \to (R^s/G_i) \otimes_R N \to 0.$$

If

$$k = \operatorname{depth}_m\bigl((G_i/G_d) \otimes_R N\bigr) < \dim(\overline{R}) = \operatorname{depth}_m \overline{N},$$

then (*) shows that

$$\operatorname{depth}_m\bigl((R^s/G_i) \otimes_R N\bigr) = k+1,$$

and then $(**)$ shows that

$$\operatorname{depth}_m\bigl(G_i/G_d) \otimes_R N\bigr) \ge k+1 > k,$$

a contradiction. If

$$\operatorname{depth}_m\bigl((R^s/G_i) \otimes_R N\bigr) < \dim(\overline{R}) - 1,$$

we get an entirely similar contradiction by first using $(**)$ and then $(*)$.

The exact sequences

$$0 \to M_i \to R^s/G_{i+1} \to R^s/G_i \to 0$$

likewise remain exact when we apply $\_ \otimes_R N$, yielding exact sequences

$$0 \to M_i \otimes_R N \to (R^s/G_{i+1}) \otimes_R N \to (R^s/G_i) \otimes_R N \to 0.$$

Since the modules in the middle and on the right are maximal Cohen-Macaulay modules over $\overline{R}$, so is $M_i \otimes_R N$, $0 \le i \le d-1$.

Since these d modules are the factors in a filtration of $(N/fN)^s$, we have that

$$e\big((N/fN)^s\big) = \sum_{i=0}^{d-1} e(M_i \otimes_R N).$$

The left hand side is $se(N/fN)$, which is $sde(N)$ by the preceding Lemma. Since there are d terms in the sum, there is at least one choice of i such that $e(M_i \otimes_R N) \le se(N)$. But

$$\nu(M_i \otimes N) = \dim_K\big(K \otimes_R (M_i \otimes_N N)\big) = \dim_K\big((K \otimes_R K) \otimes_R (M_i \otimes_R N)\big)$$

$$= \dim_K\big((K \otimes_R M_i) \otimes_K (K \otimes_R N) = \nu(M_i)\nu(N) = s\nu(N) = se(N),$$

since N is a linear maximal Cohen-Macaulay module over R. Thus, there is at least one i such that $e(M_i \otimes_R N) \le \nu(M_i \otimes_R N)$. Since the opposite inequality is automatic, for this choice of i we have that $M_i \otimes_R N$ is a linear maximal Cohen-Macaulay module over $\overline{R}$. $\square$

Corollary. *Let (R, m, K) be a local ring that is a* strict *complete intersection, i.e., the quotient of a regular ring $(T, \mathfrak{n})$ by a sequence of elements $f_1, \ldots, f_k$ whose leading forms constitute a regular sequence in $\mathrm{gr}_{\mathfrak{n}} T$. Then R has a linear maximal Cohen-Macaulay module.* $\square$

Remark. In [J. Herzog, B. Ulrich, and J. Backelin, *Linear maximal Cohen-Macaulay modules over strict complete intersections*, Journal of Pure and Applied Algebra **71** (1991) 187–202], a converse to the Theorem on p. 3 of the Lecture Notes of November 17 is obtained, showing that flitrations of R^s/fR^s like the one given by the $G_i/G)d$ all come from matrix factorizatons. Also, the authors use the fact that $I(\alpha) = I$ to prove, in certain cases, that there are infinitely many mutually non-isomorphic maximal Cohen-Macaulay modules M satisfying certain restrictions on $e(M)$ and $\nu(M)$ and, in particular, on the ratio $\dfrac{\nu(M)}{e(M)}$.

We next want to show that linear maximal Cohen-Macaulay modules exist for certain determinantal rings and for certain Segre products when both factors have linear maximal Cohen-Macaulay modules. We recall that if R and S are two finitely generated $\mathbb{N}$-graded K-algebras, the *Segre product* of R and S, which we shall denote $R \circledS_K S$, is defined as

$$\bigoplus_n R_n \otimes_K S_n,$$

which is $\mathbb{N}$-graded so that $[R \circledS_K S]_n = R_n \otimes_K S_n$. This is a K-subalgebra of the tensor product $R \otimes_K S$, which has an $\mathbb{N}^2$-grading in which

$$[R \otimes_K S]_{h,k} = R_h \otimes_K S_k.$$

Note that $R \circledS_K S$ is a direct summand of $R \otimes_K S$: an $R \circledS_K S$-module complement is

$$\bigoplus_{h \neq k} R_h \otimes_K S_k.$$

For example, if $R = K[x_1, \ldots, x_r]$ and $S = K[y_1, \ldots, y_s]$ are polynomial rings,

$$T = R \otimes_K S = K[x_1, \ldots, x_r, y_1, \ldots, y_s],$$

a polynomial ring, and $R \circledS_K S = K[x_i y_j : 1 \leq i \leq r,\ 1 \leq j \leq s] \subseteq T$. If $Z = \bigl(z_{ij}\bigr)$ is an $r \times s$ matrix of new indeterminates, the K-algebra map

$$K[z_{ij} : 1 \leq i \leq r,\ 1 \leq j \leq s] \twoheadrightarrow R \circledS_K S$$

sending $z_{ij} \mapsto x_i y_j$ can be shown to have kernel $I_2(Z)$, so that

$$K[z_{ij} : 1 \leq i \leq r, 1 \leq j \leq s]/I_2(Z) \cong R \circledS_K S.$$

See Problem 5 of Problem set #5.

We shall see eventually that the Segre product of two Cohen-Macaulay rings need not be Cohen-Macaulay in general.