

## Math 711: Lecture of November 22, 2006

Our next objective is to exhibit linear maximal Cohen-Macaulay modules for rings defined by the vanishing of the minors of a matrix of indeterminates in two special cases: one is the case of maximal minors, and the other the case of  $2 \times 2$  minors. We shall solve the second problem in two different ways, one of which generalizes to the case of Segre products of standard graded  $K$ -algebras each of which has a linear maximal Cohen-Macaulay module.

*Discussion: rings defined by the vanishing of the minors of a generic matrix.* Let  $K$  be a field and let  $X = (X_{ij})$  denote an  $r \times s$  matrix of indeterminates over  $K$ , where  $1 \leq r \leq s$ . Let  $K[X]$  denote the polynomial ring in the  $rs$  variables  $X_{ij}$ . The ideal generated by the size  $t$  minors,  $I_t(X)$ , is known to be prime: in fact,  $K[X]/I_t(X)$  is known to be a Cohen-Macaulay normal domain. This was first proved in [M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058], and was treated by two methods in the Lecture Notes from Math 711, Winter 2006: one is the method of principal radical systems, adapted from the paper just cited, and the other is via the method of Hodge algebras. We shall assume the fact that these ideals are prime here. An argument for the case  $t = 2$  is given in Problem 5 of Problem Set #5, in which the isomorphism of  $K[X]/I_2(X)$  with the Segre product of two polynomial rings over  $K$ , one in  $r$  variables and one in  $s$  variables, is established.

We note that it is easy to see that when  $K$  is algebraically closed, the algebraic set  $V(I_t(X)) \subseteq \mathbb{A}_K^{rs}$  is irreducible: this is the algebraic set of  $r \times s$  matrices of rank at most  $t - 1$ . For any such matrix  $\alpha$ , the map  $K^s \rightarrow K^r$  that it represents factors through  $K^{t-1}$ , e.g., through a  $(t - 1)$ -dimensional subspace of  $K^r$  containing the image of  $\alpha$ , and the factorization

$$K^s \rightarrow K^{t-1} \rightarrow K^r$$

enables us to write  $\alpha = \beta\gamma$  where  $\beta$  is  $r \times (t - 1)$  and  $\gamma$  is  $(t - 1) \times s$ . Any matrix that factors this way has column space contained in the column space of  $\beta$ , which shows that we have a surjection

$$\mathbb{A}_K^{r(t-1)} \times \mathbb{A}^{(t-1)s} \twoheadrightarrow V(I_t(X)).$$

This proves the irreducibility, since the image of an irreducible algebraic set is irreducible, and shows, at least, that  $\text{Rad}(I_t(X))$  is prime.

It is also easy to calculate the dimension of  $V(I_t(X))$  and, hence, of  $K[X]/I_t(X)$ . Let  $\rho = t - 1$ . Consider the open set in  $W \subseteq X$  such that the first  $\rho$  rows of  $X$  are linearly independent. The open set  $U$  of choices  $\gamma$  for these rows (we may think of points  $\gamma \in U$  as  $\rho \times s$  matrices of maximal rank) has dimension  $\rho s$ . Each remaining row is a unique linear combination of the rows of  $\gamma$  using  $\rho$  coefficients, so that the last  $r - \rho$  rows of the matrix can be written uniquely in the form  $\eta\gamma$ , where  $\eta$  is an arbitrary  $(r - \rho) \times \rho$  matrix. This

gives a bijective map of  $\mathbb{A}^{(r-\rho)\rho} \times U$  onto the dense open  $W \subseteq X$ , and so the dimension of  $V(I_t(X))$ , which is the same as  $\dim(W)$ , is  $(r-\rho)\rho + \rho s = \rho(r+s-\rho)$ , where  $\rho = t-1$ . It also follows that the height of  $I_t(X)$  is  $rs - \rho(r+s-\rho) = (r-\rho)(s-\rho)$ .

If  $t = r$ , the case of maximal minors, the height is

$$rs - (r-1)(s+1) = rs - (rs - s + r - 1) = s - r + 1.$$

If  $t = 2$ , the dimension is  $r + s - 1$ .

*Discussion: linear maximal Cohen-Macaulay modules over a standard graded ring.* Let  $K$  be a field. Recall that a *standard* graded  $K$ -algebra is a finitely generated  $\mathbb{N}$ -graded  $K$ -algebra  $R$  such that  $R_0 = K$  and  $R = K[R_1]$ , i.e.,  $R$  is generated over  $K$  by its forms of degree 1. In dealing with the existence of linear maximal Cohen-Macaulay modules over the local ring  $R_m$  of a standard graded  $K$ -algebra at its homogeneous maximal ideal  $m = \bigoplus_{n=1}^{\infty} R_n$ , it is convenient to work entirely in the graded case.

Note that  $\text{gr}_{mR_m} R_m \cong \text{gr}_m(R) \cong R$ , so that each of the local ring and the graded ring determines the other. If  $N$  is a linear maximal Cohen-Macaulay module over a local ring  $(S, \mathfrak{n}, K)$ , then  $M = \text{gr}_{\mathfrak{n}} N$  is a maximal Cohen-Macaulay module over  $R = \text{gr}_{\mathfrak{n}} S$ , and  $R$  is a standard graded  $K$ -algebra. To check this it suffices to do so after replacing  $S$  by  $S(t)$ , so that the residue class field is infinite.  $R$  is replaced by  $K(t) \otimes_K R$ , and  $M$  by  $K(t) \otimes_K M$ , which does not affect the Cohen-Macaulay property. But when the residue class field is infinite, we can choose a minimal reduction  $I = (x_1, \dots, x_r)S$  for  $m$ , where  $r = \dim(R) = \dim(S)$  and  $x_1, \dots, x_r$  is a system of parameters, and then

$$\text{gr}_{\mathfrak{n}}(N) = \text{gr}_I(N) \cong (N/IN) \otimes_K K[X_1, \dots, X_r],$$

where  $K[X_1, \dots, X_r]$  is a polynomial ring. Note that  $M = \text{gr}_{\mathfrak{n}}(N)$  is generated in degree 0:  $M_0 = N/IN = N/mN$ .

Conversely, if  $M$  is an  $\mathbb{N}$ -graded maximal Cohen-Macaulay module over the standard graded  $K$ -algebra  $R$ , and  $M$  is generated by elements of equal (necessarily smallest) degree, we shall refer to  $M$  as a *linear maximal Cohen-Macaulay module in the graded sense* over  $R$  if  $e(M) = \nu(M)$ , where  $e(M)$  is defined as the integer  $e$  such that  $\frac{e}{(r-1)!} n^{r-1}$  agrees with the leading term of the Hilbert polynomial of  $M$  (by which we mean the polynomial that agrees with  $\dim_K(M_n)$  for all  $n \gg 0$ ). Note that we can shift the grading on such an  $M$  so that it is generated in degree 0. This does not affect  $\nu(M)$  nor  $e(M)$ . This is precisely the condition for  $M_m$  to be a linear maximal Cohen-Macaulay module over  $R$ .

In fact, if we have a finitely generated graded module  $M$  over standard graded  $K$ -algebra  $R$ , then

$$\nu(M_m) = \dim_K(M_m/mM_m) = \dim_K(M/mM) = \nu(M),$$

and a minimal set of generators of  $M$  as a module may be taken to consist of homogeneous elements. Under the condition that  $M$  is generated in degree 0,  $\text{gr}_{mR_m} M_m \cong M$ ,

and  $e_{R_m}(M_m)$  may be calculated from the Hilbert function of the associated graded module, which yields that  $e_{R_m}(M_m) = e(M)$ . If  $R$  has a system of parameters  $x_1, \dots, x_r$  consisting of linear forms, which is automatic when  $K$  is infinite, then we again have  $mM = (x_1, \dots, x_r)M$  in the graded case, since  $e(M) = \ell(M/(x_1, \dots, x_r)M)$  and  $\nu(M) = \ell(M/mM)$ .

*Discussion: a linear homogeneous system of parameters for the maximal minors.* Consider an  $r \times s$  matrix  $X$  of indeterminates  $X_{i,j}$ , over  $K$ , where  $r \leq s$ . We can give a linear homogeneous system of parameters for  $K[X]/I_r(X)$  as follows. Let  $D_j$  be the diagonal whose entries are  $X_{1,j}, X_{2,j+1}, \dots, X_{r,j+r-1}$ , where  $1 \leq i \leq s - r + 1$ . The linear homogeneous system of parameters consists of the elements below these diagonals (there are  $r(r-1)/2$  such elements), the differences  $X_{k+1,j+k} - X_{j,1}$  (these are all on the diagonal  $D_j$ ) as  $j$  varies,  $1 \leq j \leq s - r + 1$ , and the elements above all the diagonals (again, there are  $r(r-1)/2$  such elements). The total number of elements is  $r(r-1) + (s-r-1)(r-1) = (r-1)(s-1)$ , which is the dimension of the ring  $K[X]/I_r(X)$ . To check that the elements specified form a system of parameters, it suffices to check that the maximal ideal is nilpotent in the quotient. Note that, because all elements below  $D_1$  are 0 and the image of  $D_1$  in the quotient has all entries equal to, say,  $x_1$  (the image of  $X_{11}$ ), we find from the vanishing of the leftmost  $r \times r$  minor that  $x_1^r = 0$ , so that  $x_1$  is nilpotent. We can then prove by induction on  $j$  that all the elements on the diagonal that is the image of  $D_j$  (these are all equal) are nilpotent. If we know this for all variables below the diagonal  $D_j$  by the induction hypothesis, and  $x_j$ , the image of  $X_{1,j}$ , is the common image of the elements on  $D_j$ , then the  $x_j^r$  is nilpotent, from the vanishing of the minor consisting of  $r$  consecutive columns beginning with the  $j$ th.

In fact, the quotient of  $K[X]/I_r(X)$  by this linear homogeneous system of parameters turns out to be isomorphic with  $K[x_1, \dots, x_{s-r+1}]/\mathcal{M}^r$ , where the numerator is a polynomial ring and  $\mathcal{M} = (x_1, \dots, x_{s-r+1})$ . This is left as an exercise: see Problem 2. in Problem Set #5. It follows that the multiplicity of the ring  $K[X]/I_r(X)$  is the number of monomials of degree at most  $r-1$  in  $s-r+1$  variables, which is  $\binom{s}{r-1}$ .

We can now show:

**Theorem.** *With notation as above, the ideal  $P$  generated by the  $r-1$  size minors of the first  $r-1$  rows of  $X$  is a linear maximal Cohen-Macaulay module for  $R = K[X]/I_r(X)$  in the graded sense.*

*Proof.* The generators of  $P$  have equal degree, and since  $P$  has rank one,  $e(P) = e(R)$ . Clearly,  $\nu(P) = \binom{s}{r-1} = e(P)$ . Thus, we need only see that  $P$  is maximal Cohen-Macaulay when considered as an  $R$ -module. The key point is that  $P$  is a height one prime in  $R$ : its inverse image in the polynomial ring has height  $s - (r-1) + 1 = s - r + 2$ , one more than the height of  $I_r(X)$ . Moreover, the quotient  $R/P$  is Cohen-Macaulay: it is a polynomial ring over a ring obtained by killing minors of an  $(r-1) \times s$  matrix of

indeterminates, and so its depth on the homogeneous maximal ideal of  $R$  is  $\dim(R) - 1$ . The short exact sequence

$$0 \rightarrow P \rightarrow R \rightarrow R/P \rightarrow 0$$

now implies that  $P$  has depth equal to  $\dim(R)$  as an  $R$ -module.  $\square$

We next want to give a calculation of the multiplicity of the ring  $R = K[X]/I_2(X)$  when  $X = (X_{ij})$  is a matrix of indeterminates. We already know from Problem 5 of Problem Set #5 and Problem 6 of Problem Set #4 that the answer is  $\binom{r+s-2}{r-1}$ . We give an alternative proof of this by a completely different method. The idea of this method is the same as the idea of the proof that these rings are Cohen-Macaulay via the technique of principal radical systems.

If  $r = 1$  the ring is a polynomial ring in  $s$  variables and the multiplicity is 1, which is correctly given by the formula. We prove that the multiplicity is  $\binom{r+s-2}{r-1}$  by induction on the number of variables. The idea is to kill one of the entries of the matrix, say  $x = x_{11}$ . Since the ring is a domain  $x_{11}$  is not a zerodivisor, and the resulting ring has the same multiplicity as  $R$ . In this ring,  $x_{1j}x_{i1} = x_{11}x_{ij} = 0$  in  $R$  for  $i, j \geq 1$ , and so every prime ideal contains either all of the elements  $x_{1j}$  or all of the elements  $x_{i1}$ . Since  $P = (x_{1j} : 1 \leq j \leq s)$  is a prime, and  $Q = (x_{i1} : 1 \leq i \leq r)$  is a prime,  $P$  and  $Q$  are precisely the minimal primes of  $x_{11}R$  in  $R$ . If we localize at  $P$  the elements  $x_{i1}$ ,  $i \geq 2$  become invertible, and the resulting ring is easily checked to be a field (localizing at  $x_{i2}$  produces a localization of a polynomial ring over  $K$ ). The situation is the same if we localize at  $Q$ . Thus,  $e(R) = e(R/P) + e(R/Q)$ . The former is the ring obtained by killing the  $2 \times 2$  minors of an  $(r-1) \times s$  matrix of indeterminates, and the latter by killing the  $2 \times 2$  minors of an  $r \times (s-1)$  matrix of indeterminates. The result now follows from the identity

$$\binom{r+s-2}{r-1} = \binom{r-1+s-2}{r-1-1} + \binom{r+s-1-2}{r-1}.$$

*Discussion: linear maximal Cohen-Macaulay modules for  $K[X]/I_2(X)$ .* Our first proof uses the fact that the ring  $K[X]/I_2(X)$  has, for  $t \geq 1$ , an endomorphism reminiscent of the Frobenius endomorphism. To wit, the  $K$ -algebra endomorphism  $K[X] \rightarrow K[X]$  that sends  $X_{ij} \mapsto X_{ij}^t$  for all  $i$  and  $j$  maps  $I_2(X)$  into itself:

$$x_{ij}x_{hk} - x_{ik}x_{hj} \mapsto x_{ij}^t x_{hk}^t - x_{ik}^t x_{hj}^t.$$

and the latter element is a multiple of the former element.

If we think of

$$R = K[X]/I_2(X) \cong K[Y_1, \dots, Y_r] \otimes_K K[Z_1, \dots, Z_s]$$

this endomorphism is induced by the  $K$ -endomorphism of the polynomial ring

$$K[Y_1, \dots, Y_r, Z_1, \dots, Z_s]$$

such that  $Y_i \mapsto Y_i^t$  and  $Z_j \mapsto Z_j^t$ . This is clearly an injective endomorphism. We can restrict this endomorphism to the Segre product. We then have  $X_i Y_j \mapsto (X_i Y_j)^t$ . From this point of view, it is clear that this endomorphism  $\theta_t$  of  $R$  is injective. We write  ${}^t R$  for  $R$  viewed as an  $R$ -module via  $\theta_t$ . The map  $R \rightarrow {}^t R$  is module-finite, since every  $x_i^t$  is in the image  $\theta_t(R)$ . A homogeneous system of parameters in  $R$  maps to a homogeneous system of parameters in  ${}^t R$ , which is Cohen-Macaulay, since  $R$  is. That is,  ${}^t R$  is a maximal Cohen-Macaulay module over  $R$ .

We can now decompose  ${}^t R$  into a large number of  $R$ -modules. It will follow that each of these, if nonzero, is a maximal Cohen-Macaulay  $R$ -module. This decomposition proceeds as follows.

We can think of  ${}^t R$  as  $R$  and the image of  $R$  as the subring  $S$  spanned by all monomials

$$Y_1^{a_1} \cdots Y_r^{a_r} Z_1^{b_1} \cdots Z_s^{b_s}$$

such that

$$a_1 + \cdots + a_r = b_1 + \cdots + b_s$$

and  $t$  divides every  $a_i$  and every  $b_j$ . Fix elements  $\underline{\alpha} = \alpha_1, \dots, \alpha_r$  and  $\underline{\beta} = \beta_1, \dots, \beta_s$  in  $\mathbb{Z}/t\mathbb{Z}$  such that

$$\alpha_1 + \cdots + \alpha_r = \beta_1 + \cdots + \beta_s$$

in  $\mathbb{Z}/t\mathbb{Z}$ . Let  $M_{\underline{\alpha}, \underline{\beta}}$  be the  $K$ -span of all monomials  $Y_1^{a_1} \cdots Y_r^{a_r} Z_1^{b_1} \cdots Z_s^{b_s}$  such that  $a_i \cong \alpha_i \pmod{t\mathbb{Z}}$ ,  $1 \leq i \leq r$  and  $b_j \cong \beta_j \pmod{t\mathbb{Z}}$ ,  $1 \leq j \leq s$ . It is easy to see that  $M_{\underline{\alpha}, \underline{\beta}}$  is an  $S$ -module.

We claim that for all  $t \geq r$ , the choice  $t-1, t-1, \dots, t-1$  for  $\alpha$  and  $0, 0, \dots, 0, t-r$  for  $\beta$  produces a linear maximal Cohen-Macaulay module, namely  $M_{\underline{\alpha}, \underline{\beta}}$ , in the graded sense. This module has rank one, because multiplication by  $Y_1 \cdots Y_r Z_s^r$  produces an ideal in  $S$ . Hence, the multiplicity is the same as for  $S$ , i.e.,  $\binom{r+s-2}{r-1}$ . It is easy to see that this module is generated minimally by all monomials of the form

$$Y_1^{t-1} \cdots Y_r^{t-1} Z_1^{a_1 t} \cdots Z_{s-1}^{a_{s-1} t} Z_s^{t-r},$$

where the  $a_i$  are nonnegative and

$$a_1 + \cdots + a_{s-1} = r - 1.$$

These generators all have the same degree, and the number of generators is the same as the number of monomials of degree  $r-1$  in  $s-1$  variables, which is  $\binom{r+s-2}{r-1}$ , as required.  $\square$