Math 711: Lecture of November 27, 2006

Before giving a second proof of the existence of linear maximal Cohen-Macaulay modules for $K[X]/I_2(X)$ and the extension of this result to the case of more general Segre products, we want to note that the Segre product of two Cohen-Macaulay rings need not be Cohen-Macaulay, even when one of them is a normal hypersurface and the other is a polynomial ring.

In fact, let K be any field whose characteristic is different from 3, and let

$$R = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z]$$

and $S = K[s, t]$ where X, Y, Z, s, and t are indeterminates over K. We shall show that $T = R \circledS_K S$ is a three-dimensional domain that is not Cohen-Macaulay.

We have that

$$T = K[xs, ys, zs, xt, yt, zt] \subseteq K[x, y, z, s, t].$$

The equations

$$(zs)^3 + \bigl((xs)^3 + (ys)^3\bigr) = 0 \quad \text{and} \quad (zt)^3 + \bigl((xt)^3 + (yt)^3\bigr) = 0$$

show that zs and zt are both integral over $D = K[xs, ys, xt, zt] \subseteq T$. The elements x, y, s, and t are algebraically independent, and the fraction field of D is $K[xs, ys, t/s]$, so that $\dim(D) = 3$, and

$$D \cong K[X_{11}, X_{12}, X_{21}, X_{22}]/(X_{11}X_{22} - X_{12}X_{21})$$

with $X_{11}, X_{12}, X_{21}, X_{22}$ mapping to xs, ys, xt, yt respectively.

It is then easy to see that $ys, xt, xs - yt$ is a homogeneous system of parameters for D, and, consequently, for T as well. The relation

$$(zs)(zt)(xs - yt) = (zs)^2(xt) - (zt)^2(ys)$$

now shows that T is *not* Cohen-Macaulay, for $(zs)(zt) \notin (xt, ys)T$. To see this, suppose otherwise. The map

$$K[x, y, z, s, t] \to K[x, y, z]$$

that fixes $K[x, y, z]$ while sending $s \mapsto 1$ and $t \mapsto 1$ restricts to give a K-algebra map

$$K[xs, ys, zs, xt, yt, zt] \to K[x, y, z].$$

If $(zs)(zt) \in (xt, ys)T$, applying this map gives $z^2 \in (x, y)K[x, y, z]$, which is false — in fact, $K[x, y, z]/(x, y) \cong K[z]/(z^3)$. $\square$

Segre products do have good properties that are important. It was already noted that $R \circledS_K S$ is a direct summand of $R \otimes_K S$. This implies that every ideal of $R \circledS_K S$ is contracted from $R \otimes_K S$. In particular, $R \circledS_K S$ is Noetherian and, since it is $\mathbb{N}$-graded, finitely generated over K. This is quite obvious when R and S are standard, since it is then generated by the products of elements in a basis for R_1 with elements in a basis for S_1. When $R \otimes_K S$ is normal, so is $R \circledS_K S$. In particular, this is true of the ring in the example above. $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ is normal, since it is Cohen-Macaulay and the singular locus is the origin (the partial derivatives of $X^3 + Y^3 + Z^3$ vanish simultaneously only at the origin), and $R \otimes_K S = R[s, t]$.

Discussion: the dimension of the Segre product. For any finitely generated N-graded K-algebras R and S with $R_0 = K = S_0$, we have that

$$\dim (R \circledS_K S) = \dim (R) + \dim (S) - 1.$$

Each of R and S has a homogeneous system of parameters. After raising the elments to powers, we find that R is module-finite over $A = K[F_1, \ldots, F_r]$, where $F_1, \ldots, F_r$ form a homogeneous system of parameters of degree k, and S is module-finite over $B = K[G_1, \ldots, G_s]$ where $G_1, \ldots, G_s$ is a homogeneous system of parameters of degree k as well. If K is infinite and R and S are standard, we may even assume that $k = 1$ here. Then $A \circledS_K B \cong K[X]/I_2(X)$ where X is an $r \times s$ matrix of indeterminates over K, and so has dimension $r + s - 1$. The result now follows because $R \circledS_K S$ is module-finite over $A \circledS_K B$. To see this, choose $h \gg 0$ so that homogenous generators for R over A and for S over B have degree $\leq h$. Let V_k be a K-basis for R_k for $k \leq h$ and let W_k be a K-basis for S_k for $k \leq h$. Then the finite set $\mathcal{S}$ of elements of the form $v \otimes w$, where $v \in V_k$ and $W \in w_k$ for some $k \leq h$, generate $R \circledS_K S$ *as a module* over $A \circledS_K B$. To see this, let $F \in R_t$ and $G \in S_t$ be given. Then F is an A-linear combination of elements in a fixed V_k with coefficients in A_{t-k}, and G is a B-linear combination of elements in a fixed W_k with coefficients in B_{t-k}. Here, if $t \leq h$ one may take $k = t$, and if $t \geq h$, one may take $k = h$. It follows that every element of the form $F \otimes G$ is in the $A \circledS_K B$-span of $\mathcal{S}$, as claimed, and elements of this form span $R \circledS_K S$ over K. $\square$

Our next objective is to give a different proof that $R = K[X]/I_2(X)$ has a linear maximal Cohen-Macaulay module. Again, we consider the isomorphism

$$K[X]/I_2(X) \cong S = K[Y_1, \ldots, Y_r] \circledS_K K[Z_1, \ldots, Z_s] \subseteq KY_1, \ldots, Y_r, Z_1, \ldots, Z_s] = T.$$

For every $\delta \in \mathbb{Z}$, let T_δ denote the K-span of the monomials $\mu \in T$ such that

$$\deg_Y(\mu) - \deg_Z(\mu) = \delta,$$

where $\deg_Y(\mu)$ denotes the total degree of μ in the variables $Y_1, \ldots, Y_r$ and $\deg_Z \mu$ denotes the total degree of μ in the variables $Z_1, \ldots, Z_s$. Then T_δ is obviously an S-module, and $T = \bigoplus_{\delta \in \mathbb{Z}} T_\delta$.

The following result is proved in [S. Goto and K.-i. Watanabe, *On graded rings*, I, Journal of the Mathematical Society of Japan **30** (1978) 179–213].

Theorem. *With the above notation, T_δ is a maximal Cohen-Macaulay module of torsion-free rank one over the Segre product*

$$K[Y_1, \ldots, Y_r] \circledS_K K[Z_1, \ldots, Z_s] = S \cong R = K[X]/I_2(X)$$

for $s > \delta > -r$.

Proof. The case where $\delta = 0$ is the statement that $T_0 = S$ is Cohen-Macaulay, which we are assuming here. We assume that $\delta \geq 0$ and proceed by induction on s. The case where $0 \geq \delta > -r$ then follows by interchanging the roles of $Y_1, \ldots, Y_r$ and $Z_1, \ldots, Z_s$.

The case where $s = 1$ is obvious. Note that $T_\delta \cong Z_1^\delta T_\delta$, and that $Z_1^\delta T_\delta$ is the ideal generated by the monomials of degree δ in $Y_1Z_1, \ldots, Y_rZ_1$, which is P^δ, where $P = (Y_1Z_1, \ldots, Y_rZ_1)$. P corresponds to the prime ideal of R generated by the variables in the first column. The quotient is $K[X^-]/I_2(X^-)$, where X^- is the $r \times (s-1)$ matrix obtained by omitting the first column of X: this ring has dimension $r + (s-1) - 1 = r + s - 2$, from which it follows that P is a height one prime of S. To complete the proof, it will suffice to show that for $1 \leq \delta < s$, S/P^δ is Cohen-Macaulay: the short exact sequence

$$0 \to P^\delta \to S \to S/p^\delta \to 0$$

then shows that

$$\operatorname{depth}_m P^\delta = \operatorname{depth}_m (S/P^\delta) + 1 = \dim (S).$$

We filter S/P^δ by the modules P^k/P^{k+1}, $0 \leq k < \delta$. Each of these is a module over S/P, and it suffices to show that each is a maximal Cohen-Macaulay module over S/P. We already know this when $k = 0$, and so we may assume that $1 \leq k < \delta$.

We make use of the fact for every $k \in \mathbb{N}$, $P^k = Z_1^k T \cap S$. This gives an injection of

$$P^k/P^{k+1} \hookrightarrow Z_1^k T/Z_1^{k+1} T \cong T/Z_1 T \cong K[Y_1, \ldots, Y_r, Z_2, \ldots, Z_s] = T^-.$$

If we identify P^k/P^{k+1} as a submodule of T^- in this way, the action of S/P is obtained by identifying $S/P \cong K[Y_1, \ldots, Y_r] \circledS_K K[Z_2, \ldots, Z_s] \subseteq T^-$. The generators of P^k map to the monomials of degree k in $Y_1, \ldots, Y_r$. Thus, we may identify P^k/P^{k+1} with T_k^-. Since s has been decreased by 1 and $k < \delta \leq s - 1$, the result follows from the induction hypothesis. $\square$

Corollary. *With notation as in the Theorem above, T_{s-1} is a linear maximal Cohen-Macaulay module over $S \cong R$.*

Proof. The generators of T_{s-1} have the same degree, and since T_{s-1} is rank one,

$$e(T_{s-1}) = e(T_0) = \binom{r+s-2}{r-1},$$

which is the same as the number of monomials of degree $s-1$ in $Y_1, \ldots, Y_r$. $\square$

We can now prove:

Theorem (D. Hanes). *Let K be an infinite field. Let R and S be standard graded K-algebras that possess linear maximal Cohen-Macaulay modules in the graded sense. Then so does $R \circledS_K S$.*

Proof. We may assume that M and N are the linear maximal Cohen-Macaulay modules over R and S respectively and that they are generated in degree 0. Let $Y_1, \ldots, Y_r$ be a homogeneous linear system of parameters for R, where $r = \dim(R)$, and let $Z_1, \ldots, Z_s$ be a homogenous linear system of parameters for S, where $s = \dim(S)$. Let m and $\mathfrak{n}$ be the respective homogeneous maximal ideals in R and S.

Then R is module-finite over $A = K[Y_1, \ldots, Y_r]$, and S is module-finite over $B = K[Z_1, \ldots, Z_s]$. Moreover, $R \otimes_K S$ is module-finite over $A \otimes_K B$, and $R \circledS_K S$ is module-finite over $A \circledS_K B$, by the Discussion on the dimension of Segre products. Since M is Cohen-Macaulay it is A-free, and its rank $c = e(M)$. Similarly, N is B-free of rank $d = e(N)$. Note that $mM = (Y_1, \ldots, Y_r)M$ and that $\mathfrak{n}N = (Z_1, \ldots, Z_s)N$.

The action of any degree one form F of R on $M \cong A^c$ is an A-linear map and can be thought of as being given by a $c \times c$ matrix over A. Since multiplication by F increases degrees by one, the entries of each such matrix must be degree one forms of A. Similarly, the action of any degree one form $G \in S$ on N is given by a $d \times d$ matrix of linear forms over B.

Consider the $R \otimes_K S$ module $M \otimes_K N$. We can consider it as

$$A^c \otimes_K B^d \cong T^{cd},$$

where

$$T = K[Y_1, \ldots, Y_r] \otimes_K K[Z_1, \ldots, Z_s].$$

For $\delta \in \mathbb{Z}$, we can define $(M \otimes_K N)_\delta$ as $(T_\delta)^{cd}$. Because the action of forms of R and forms of S preserves the bigrading on T^{cd} coming from the Y-grading and the Z-grading on T, every $(M \otimes_K N)_\delta$ is a module over $R \circledS_K S$. Since $(M \otimes_K N)_\delta$ is finitely generated even as a module over

$$T_0 = A \circledS_K B \subseteq R \circledS_K S,$$

it is finitely generated over $R \circledS_S$. For $s > \delta > -r$ it is maximal Cohen-Macaulay over $R \circledS_K S$, since $R \circledS_K S$ is module-finite over $A \circledS_K B$, and we know that it is maximal Cohen-Macaulay over $A \circledS_K B$.

To complete the proof, we shall show that $W = (M \otimes_K N)_{s-1}$ is a linear maximal Cohen-Macaulay module over $R \circledS_K S$. It is maximal Cohen-Macaulay and generated by elements of equal degree. To complete the argument, we shall prove that

$$\nu(W) = cd \binom{r+s-2}{r-1} = e(W).$$

Include $Y_1, \ldots, Y_r$ in a set of one-forms $F_1, \ldots, F_h$ that generate m, and include $Z_1, \ldots, Z_s$ in a set of one-forms $G_1, \ldots, G_k$ that generate $\mathfrak{n}$. Let $\mathcal{M}$ be the maximal ideal of $R \circledS_K S$, which is generated by the products F_iG_j. Let $\mathcal{Q}$ be the maximal ideal of $A \circledS_K B$, which is generated by the products Y_iZ_j. Then for every integer $n \geq 0$,

$$\mathcal{M}^n(M \otimes_K N) = (F_iG_j : 1 \leq i \leq h,\, 1 \leq j \leq k)^n(M \otimes_K N) =$$

$$(F_i : 1 \leq i \leq r)^n(G_j : 1 \leq j \leq s)^n(M \otimes_K N) =$$

$$\big((F_i : 1 \leq i \leq h)^n M\big) \otimes_K \big((G_j : 1 \leq j \leq k)^n N\big) =$$

$$m^n M \otimes_K \mathfrak{n}^n N = \big((Y_i : 1 \leq i \leq r)^n M\big) \otimes_K \big((Z_j : 1 \leq j \leq s)^n N\big) =$$

$$\big((Y_i : 1 \leq i \leq r)(Z_j : 1 \leq j \leq s)\big)^n (M \otimes_K N\big) = \mathcal{Q}^n(M \otimes_K N).$$

Since $M \otimes_K N$ splits into

$$\bigoplus_{\delta \in \mathbb{Z}} (M \otimes_K N)_\delta$$

as $R \circledS_K S$-modules, we also have that

$$\mathcal{M}^n(M \otimes_K N)_\delta = \mathcal{Q}^n(M \otimes_K N)_\delta$$

for every δ. In particular, $\mathcal{M}^n W = \mathcal{Q}^n W$ for all n. Consequently, we have that $\nu(W) = \ell(W/\mathcal{M}W) = \ell(W/\mathcal{Q}W)$, which is the number of generators of W as a module over $A \circledS_K B$. Since W is the direct sum of cd copies of $(A \circledS_K B)_{s-1}$, this is $cd\binom{s+r-2}{r-1}$, as required. Similarly,

$$\dim{}_K(\mathcal{M}^n W/\mathcal{M}^{n+1}W) = \dim{}_K(\mathcal{Q}^n W/\mathcal{Q}^{n+1}W),$$

and this is cd times the Hilbert function of $(A \circledS_K B)_{s-1}$ with respect to $\mathcal{Q}$. The multiplicity is therefore

$$cd\, e_{\mathcal{Q}}\big((A \circledS_K B)_{s-1}\big) = cd\binom{r+s-2}{r-1}. \qquad \square$$