Math 711: Lecture of December 1, 2006

Recall the if R is a Noetherian ring of prime characteristic $p > 0$, R is called F-*finite* if $F : R \to R$ makes R into a module-finite algebra over

$$F(R) = \{r^p : r \in R\},$$

a subring of R that is also denoted R^p. When R is F-finite, the composition $F^e : R \to R$ also makes R into a finite module over

$$F^e(R) = \{r^{p^e} : r \in R\},$$

a subring of R that is alternatively denoted R^{p^e}.

If R is F-finite, it is trivial that every homomorphic image of R is F-finite. The same holds for each localization $W^{-1}R$, because inverting the elements in W^p has the effect of inverting the elements of W. If R is F-finite, so is $R[x]$: if $r_1, \ldots, r_h$ span R over $F(R)$, then the elements $r_i x_j$, $1 \le i \le h$, $0 \le j < p$ span $R[x]$ over $F(R[x]) = F(R)[x^p]$. By induction, any finitely generated algebra over an F-finite ring is F-finite, and it is likewise true that any algebra essentially of finite type over an F-finite ring is F-finite.

A perfect field is obviously F-finite, and so a field that is finitely generated as a field over a perfect field is F-finite: it is a localization of a finitely generated algebra over a perfect field. Thus, if K is perfect, each of the fields $K(t_1, \ldots, t_n)$ is F-finite, where $t_1, t_2, \ldots, t_n, \ldots$ are indeterminates over K, but the field $K(t_1, \ldots, t_n, \ldots)$ where we adjoin infinitely many indeterminates, is not. We note:

Proposition. *A complete local ring (R, m, K) of prime characteristic $p > 0$ is F-finite if and only if its residue class field K is F-finite.*

Proof. Since $K = R/m$, if R is F-finite then K is. Suppose that K is F-finite, and let $c_1, \ldots, c_h$ be a basis for K over $F(K)$. Then R is a homomorphic image of a formal power series ring $S = K[[x_1, \ldots, x_d]]$, and it suffices to show that S is F-finite. But the set of elements

$$\{c_j x_1^{a_1} \cdots x_d^{a_d} : 0 \le j \le h,\ 0 \le a_i < p \text{ for } 0 \le i \le d\}$$

spans S over $F(S) = F(K)[[x_1^p, \ldots, x_d^p]]$. $\square$

This justifies the assertion in the Lecture Notes of November 29 that a complete local ring with perfect residue class field is F-finite.

We next want to understand the behavior of the rank of ${}^eR$ when R is a complete local domain with a perfect residue class field.

Note that when R is reduced of prime characteristic $p > 0$, the three maps $F^e : R \to R$, $R^{p^e} \subseteq R$, and $R \subseteq R^{1/p^e}$ are isomorphic. The isomorphism of $F^e : R \to R$ with $R^{p^e} \subseteq R$

follows from the fact that, for a reduced ring R, F^e is injective and $F^e(R) = R^{p^e}$. To understand the third map, we need to define the ring R^{1/p^e}. When R is a domain, there we may take this to be the subring of an algebraic closure of the fraction field of R that consists of all the elements of the for r^{1/p^e} for $r \in R$. In the general case, one can show that there is an extension S of R, unique up to canonical isomorphism, such that the map $R = \{s^{p^e} : s \in S\}$. In fact, since $R \cong R^{p^e}$ via the map $r \mapsto r^{p^e}$, we "think of" R^{p^e} as R, and take S to be R.

This means that when R is reduced, we may think of ${}^eR$ as R^{1/p^e}.

Theorem. *Let (R, m, K) be a complete local ring of Krull dimension d such that K is perfect. Then for every $e \in \mathbb{N}$, the torsion-free rank of ${}^eR$ over R is p^{de}.*

Proof. By the structure theory of complete local rings, R is module finite over $A = K[[x_1, \dots, x_d]]$. Let $\mathrm{frac}\,(R) = \mathcal{L}$ and $\mathrm{frac}\,(A) = \mathcal{K}$. The torsion free rank of R^{1/p^e} over R is the same as $[\mathcal{L}^{1/p^e} : \mathcal{L}]$. We have that

$$[\mathcal{L}^{1/p^e} : \mathcal{K}] = [\mathcal{L}^{1/p^e} : \mathcal{K}^{1/p^e}]\,[\mathcal{K}^{1/p^e} : \mathcal{K}]$$

and also

$$[\mathcal{L}^{1/p^e} : \mathcal{K}] = [\mathcal{L}^{1/p^e} : \mathcal{L}]\,[\mathcal{L} : \mathcal{K}],$$

so that

$$(*) \quad [\mathcal{L}^{1/p^e} : \mathcal{K}^{1/p^e}]\,[\mathcal{K}^{1/p^e} : \mathcal{K}] = [\mathcal{L}^{1/p^e} : \mathcal{L}]\,[\mathcal{L} : \mathcal{K}].$$

The map $u \to u^{1/p^e}$ gives an isomorphism of the inclusion $\mathcal{K} \subseteq \mathcal{L}$ with the inclusion $\mathcal{K}^{1/p^e} \subseteq \mathcal{L}^{1/p^e}$, so that

$$[\mathcal{L}^{1/p^e} : \mathcal{K}^{1/p^e}] = [\mathcal{L} : \mathcal{K}].$$

But then $(*)$ implies that

$$[\mathcal{L}^{1/p^e} : \mathcal{L}] = [\mathcal{K}^{1/p^e} : \mathcal{K}],$$

and the latter is the same as the torsion-free rank over A of

$$B = A^{1/p^e} \cong K[[x_1^{1/p^e}, \dots, x_d^{1/p^e}]].$$

Let $y_i = x_i^{1/p^e}$, $1 \le i \le d$. Then B is free over A on the basis consisting of all monomials $y_1^{a_1} \cdots y_d^{a_d}$ with $0 \le a_i < p^e$ for $1 \le i \le d$. This free basis has cardinality $(p^e)^d = p^{de}$, as required. $\square$

We are now ready to prove the existence of Hilbert-Kunz multiplicities: the result is stated on the first page of the Lecture Notes of November 29, but we repeat the statement.

Theorem (Monsky). *Let M be a finitely generated module of dimension d over (R, m, K), where R has prime characteristic $p > 0$, and let $\mathfrak{A} \subseteq m$ be m-primary. Then the Hilbert-Kunz multiplicity $e_{HK}(\mathfrak{A}, M)$ of M with respect to $\mathfrak{A}$ exists, and is a positive real number.*

Proof. By the results of the Lecture of November 29, it suffices to prove this when $M = (R,\, m,\, K)$ is a complete local domain with a perfect residue class field. Let

$$\gamma_n = \frac{\ell(R/\mathfrak{A}^{[p^n]})}{p^{nd}}.$$

We shall prove that the sequence $\{\gamma_n\}_n$ is a Cauchy sequence. This will prove that the sequence has a limit. The fact that the limit is positive then follows from the lower bound in part (c) of the Lemma on p. 2 of the Lecture Notes of November 29.

The first key point is that ${}^1R \cong R^{1/p}$ has torsion-free rank p^d as an R-module. Thus, ${}^1R$ and $R^{\oplus p^d}$ become isomorphic after localization at a nonzero element of the domain R. By part (b) of the Lemma on p. 4 of the Lecture Notes of November 29, there is a positive real constant C such that

$$(*) \quad |\mathcal{F}_{HK}(\mathfrak{A},\, R^{\oplus p^d})(n) - \mathcal{F}_{HK}(\mathfrak{A},\, {}^1R)(n)| \le C/p^{(d-1)n}$$

for all $n \in \mathbb{N}$. The leftmost term is $p^d\mathcal{F}_{HK}(\mathfrak{A},\, R)(n)$. By the Proposition at the top of p. 4 of the Lecture Notes of November 29,

$$\mathcal{F}_{HK}(\mathfrak{A},\, {}^1R)(n) = \mathcal{F}_{HK}(\mathfrak{A},\, R)(n+1).$$

Thus, $(*)$ becomes

$$(**) \quad |p^d\mathcal{F}_{HK}(\mathfrak{A},\, R)(n) - \mathcal{F}_{HK}(\mathfrak{A},\, R)(n+1)| \le Cp^{(d-1)n}.$$

We may divide both sides by $p^{(n+1)d}$ to obtain

$$(***) \quad |\gamma_n - \gamma_{n+1}| \le C/p^{dn-n-dn-d} = \frac{Cp^{-d}}{p^n}.$$

Hence, for all $N \ge n$,

$$|\gamma_n - \gamma_N| \le |\gamma_n - \gamma_{n+1}| + |\gamma_{n+1} - \gamma_{n+2}| + \cdots + |\gamma_{N-1} - \gamma_N|$$
$$\le \frac{Cp^{-d}}{p^n}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots) \le \frac{Cp^{-d}(1-1/p)^{-1}}{p^n},$$

which shows that $\{\gamma_n\}_n$ is a Cauchy sequence, as claimed. $\square$

The proof of the Theorem of Monsky can be easily adapted to show more.

Theorem. *Let $(R,\, m,\, K)$ by a local ring of prime characteristic $p > 0$, let M be a finitely generated R-module, and let $\mathfrak{A} \subseteq m$ be an m-primary ideal. Then the Hilbert-Kunz multiplicity with respect to $\mathfrak{A}$ is additive in the sense that if one has a finite filtration of M with factors N_i, $e_{HK}(\mathfrak{A},\, M)$ is the sum of the values of $e_{HK}(\mathfrak{A},\, N_i)$ for those N_i of the same dimension as M.*

Equivalently, if $\mathcal{P}$ is the set of (necessarily minimal) primes in the support of M such that $\dim(R/P) = \dim(M)$*, then*

$$(*) \quad e_{HK}(\mathfrak{A},\, M) = \sum_{P \in \mathcal{P}} \ell_{R_P}(M_P) e_{HK}(\mathfrak{A},\, R/P).$$

Proof. Additivity implies the formula $(*)$ because if one takes a prime cyclic filtration of M, the only terms that contribute to the value of $e_{HK}(\mathfrak{A},\, M)$ are those R/P with $P \in \mathcal{P}$, and the number of factors equal to R/P is the same as $\ell_{R_P}(M_P)$. On the other hand, it is easy to see that if one has $(*)$, then additivity follows because for every $P \in \mathcal{P}$, $\ell_{R_P}(N_P)$ is additive in N for modules N whose support is contained in $\mathrm{Supp}\,(M)$.

It will suffice to prove additivity after applying $S \otimes_R \_$, where $(S,\, \mathfrak{n},\, L)$ is complete local with L algebraically closed, $R \to S$ is flat local, and $\mathfrak{n} = mS$. Hence, we may assume without loss of generality that R is complete local with algebraically closed residue class field, and it suffices to prove that $(*)$ holds in this case.

We may replace M by M/N where N is a maximal submodule of smaller dimension without affecting the issue. Thus, we may assume without loss of generality that M is of pure dimension. We may replace R by $R/\mathrm{Ann}_R M$ without affecting any relevant issue, so that the minimal primes of the support of M are those of R.

Let W be the multiplicative system that is the complement of the union of the minimal primes of R. Exactly as in the argument on pages 5 and 6 of the Lecture Notes of November 29, we have $M_1 \oplus \cdots \oplus M_h \subseteq M$ such that each M_i has a unique minimal prime $P_i \in \mathcal{P}$ in its support and localization at W induces an isomorphism. We then have that

$$\ell_{R_{P_i}}(M_{P_i}) = \ell_{R_{P_i}}\big((M_i)_{P_i}\big)$$

for every i while $(M_i)_{P_j} = 0$ if $j \neq i$. We then have that

$$e_{HK}(\mathfrak{A},\, M) = e_{HK}(\mathfrak{A},\, M_1 \oplus \cdots \oplus M_h) = \sum_{i=1}^{h} e_{HK}(\mathfrak{A},\, M_i),$$

and the formula $(*)$ will follow if we can show that it holds for all of the M_i. We have thus reduced to the case where M has a unique minimal prime P in its support.

If we repalce M by ${}^eM$, the Hilbert-Kunz multiplicity with respect to $\mathfrak{A}$ is multiplied by p^{de}, by the Proposition on p. 4 of the Lecture Notes of November 29. The same is true for $\ell_{R_P}(M)$: if M_P has a filtration by k copies of $\kappa = R_P/PR_P$, $({}^eM)_P$ has a filtration by k copies of ${}^e\kappa$, and the dimension of ${}^e\kappa$ over κ is the same as the torsion-free rank of ${}^e(R/P)$ over R/P, which is p^{de}, as required, by the first Theorem on p. 2 of today's Lecture Notes.

Thus, we may replace M by ${}^eM$ for $e \gg 0$, and so reduce to the case where $R/\mathrm{Ann}_R M$ is a domain. Hence, we can reduce to the case where R is a domain and M is torsion-free over R. If M has torsion-free rank ρ over R, we have already seen that $e_{HK}(\mathfrak{A},\, M) = \rho\, e_{HK}(\mathfrak{A},\, R)$, which is just what we need. $\square$

Example. To illustrate how complicated the behavior of Hilbert-Kunz multiplicities can be in relatively simple examples, we consider

$$R = \mathbb{Z}_5[[X_1,\, X_2,\, X_3,\, X_4]](X_1^4 + X_2^4 + X_3^4 + X_4^4]].$$

It is proved in [C. Han and P. Monsky, *Some surprising Hilbert-Kunz functions*, that

$$\mathcal{F}_{HK}(R)(n) = \frac{168}{61}(5^n)^3 - \frac{107}{61}3^n.$$

In particular, $e_{HK}(R) = \dfrac{168}{61}$.

Discussion. Hilbert-Kunz multiplicities can be used to characterize tight closure in complete local domains. This characterizes tight closure many instances. If R is essentially of finite type over an excellent local ring, $r \in R$ is in the tight closure of the ideal I if and only if for every complete local domain D to which R maps, the image of r is in the tight closure of ID. See Theorem (2.1) of [M. Hochster, *Tight closure in equal characteristic, big Cohen-Macaulay algebras, and solid closure*, in Commutative Algebra: Syzygies, Multiplicities and Birational Algebra, Contemp. Math. **159**, Amer. Math. Soc., Providence, R. I., 1994, 173–196], which gives a summary of many properties of tight closure. Moreover, in an excellent local ring $(R,\, m,\, K)$, $r \in R$ is in the tight closure of I if and only if it is in the tight closure of every m-primary ideal containing I.[1]

Therefore, much of the theory can be developed from a criterion for when an element $r \in R$ is in the tight closure of an m-primary ideal I in a complete local domain R. In this situation, one such criterion is the following: for a proof see Theorem (8.17) of [M. Hochster and C. Huneke, *Tight closure, invariant theory, and the generic perfection of determinantal loci*, Journal of the Amer. Math. Soc. **3** (1990) 31–116].

Theorem. *Let $(R,\, m,\, K)$ be a complete local domain of prime characteristic $p > 0$, let I be an m-primary ideal of R, let $r \in m$, and let $J = I + rR$. Then r is in the tight closure of I in R if and only $e_{HK}(I,\, R) = e_{HK}(J,\, R)$.*

Time permitting, we still aim to prove two results of Lech: one is that his conjecture holds when the base ring has dimension 2, and the other is that it holds in equal characteristic when the closed fiber S/mS of the map $(R,\, m,\, K) \to (S,\, \mathfrak{n},\, L)$ is a complete intersection. Cf. [C. Lech, *Note on multiplicities of ideals*, Arkiv for Mathemtik **4** (1960) 63–86].

However, we shall first focus on some results of D. Hanes in positive characteristic, including the fact that Lech's conjecture holds for graded rings of dimension 3 with a perfect residue class field, which is proved by means of the construction of "approximately" linear maximal Cohen-Macaulay modules.

[1] Necessity is obvious. For sufficiency, one may pass to the reduced case and then R has a test element c (see Theorem (6.1) of [M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994) 1–62]). If $cu^q \notin I^{[q]}$, we can choose N so large that $cu^q \notin I^{[q]} + m^N$. Then $cu^q \notin (I + m^N)^{[q]}$, and so u is not in the tight closure of $I + m^N$. $\square$