Math 711: Lecture of December 4, 2006

The next two results suggest that characteristic p techniques may be helpful in proving the existence of linear maximal Cohen-Macaulay modules.

Let R be a ring of prime characteristic $p > 0$. The *Frobenius closure* I^{F} of an ideal $I \subseteq R$ is

$$\{r \in R : \text{for some } e \in \mathbb{N},\ r^{p^e} \in I^{[p^e]}\}.$$

Note that once this holds for one value of e, it also holds for all larger values. Alternatively, I^{F} is the union of contractions of I to R under the maps $F^e : R \to R$ as e varies: the union is increasing. Note that $I \subseteq I^{\mathrm{F}}$. When R is Noetherian, the contractions of I under the various F^e are the same for all $e \gg 0$. Thus, if $J = I^{\mathrm{F}}$, we can choose $e \gg 0$ such that $J^{[p^e]} \subseteq I^{[p^e]}$. But since $I \subseteq J$, the opposite inclusion is obvious. Hence, for all $e \gg 0$, $(I^{\mathrm{F}})^{[p^e]} = I^{[p^e]}$. Notice that when $r \in I^{\mathrm{F}}$, we have that $1 \cdot r^{p^e} \in I^{[p^e]}$ for all $e \gg 0$, so that $I^{\mathrm{F}} \subseteq I^*$, the tight closure of I in R.

Theorem (D. Hanes). *Let (R, m, K) be an F-finite Cohen-Macaulay local ring of prime characterisitc $p > 0$. Suppose that there exists an ideal $I \subseteq m$ generated by a system of parameters such that $I^{\mathrm{F}} = m$. Then for all sufficiently large e, ${}^eR$ is a linear maximal Cohen-Macaulay module over R.*

Proof. Choose any e such that $m^{[p^e]} = I^{[p^e]}$. Since R is F-finite, ${}^eR$ is a finitely generated module over R, and, obviously a maximal Cohen-Macaulay module: if $x_1, \ldots, x_d$ is a system of parameters in R, they form an R-sequence on ${}^eR$ because $x_1^{p^e}, \ldots, x_d^{p^e}$ is a regular sequence on R. Note that under the identification of ${}^eR$ with R, $I\,{}^eR$ becomes $I^{[p^e]}$ and $m\,{}^eR$ becomes $m^{[p^e]}$. Since $I^{[p^e]} = m^{[p^e]}$, we have that $I\,{}^eR = m\,{}^eR$, as required. $\square$

Theorem. *Let (R, m, K) be any F-finite Cohen-Macaulay ring of prime characteristic $p > 0$. Then R has a free extension S such that $R \to S$ is local, the induced map of residue class fields is an isomorphism, and S has a linear maximal Cohen-Macaulay module.*

Proof. Let $x_1, \ldots, x_d$ be any system of parameters for R. Then for any sufficiently large integer $e \in \mathbb{N}$, we have that $m^{[p^e]} \subseteq I$. Let $Z_1, \ldots, Z_d$ be indeterminates over R, let $T = R[Z_1, \ldots, Z_d]$, and let $S = T/J$, where J is generated by the elements $Z_i^{p^e} - x_i$, $1 \leq i \leq d$. Evidently, S is module-finite over R, and so its maximal ideals all lie over m. But

$$S/mS \cong T/(Z_i^{p^e} : 1 \leq i \leq d)T,$$

a zero-dimensional local ring with residue class field isomorphic with K. Thus, S is local with residue class field K. Evidently, S is free over R on the basis consisting of the images of all monomials $Z_1^{a_1} \cdots Z_d^{a_d}$ with $0 \leq a_i \leq p^e$ for $1 \leq i \leq d$. Thus, S satisfies all of the requirements of the Theorem, provided that we can show that it has a linear maximal Cohen-Macaulay module.

Let $z_1, \dots, z_d$ be the images of $Z_1, \dots, Z_d$, respectively, in S. Clearly, $z_1, \dots, z_d$ is a system of parameters for S, since killing them produces $R/(x_1, \dots, x_d)R$. Since S is free over the Cohen-Macaulay ring R, it is Cohen-Macaulay. It will therefore suffice to show that it satisfies the hypothesis of the preceding Theorem. In fact, the maximal ideal $\mathfrak{n}$ of S is the Frobenious closure of $(z_1, \dots, z_d)S$. The ideal $\mathfrak{n}$ is generated by m and the z_i. But

$$m^{[p^e]} \subseteq (x_1, \dots, x_d) \subseteq (z_1, \dots, z_d)^{[p^e]}$$

since $x_i = z_i^{p^e}$ in S, while it is obvious that every $z_i^{p^e} \in (z_1, \dots, z_d)^{[p^e]}$. $\square$

We next want to discuss some results concerning the existence of linear maximal Cohen-Macaulay modules over Veronese subrings of polynomial rings.

This problem may seem rather special, but the ideas used to solve the problem in dimension three, for example, can be used to prove the existence of "approximately linear" maximal Cohen-Macaulay modules for standard graded domains over a perfect field of positive characteristic in dimension 3, and this circle of ideas has provided a substantial body of results on Lech's conjecture for standard graded algebras.

Let K be field and let S be a standard graded K-algebra. By the t th Veronese subring $S^{(t)}$ of S we mean

$$\bigoplus_{i=0}^{\infty} S_{it},$$

which may also be described as the K-algebra $K[S_t]$ generated by S_t. Clearly, S is module-finite over $S^{(t)}$, since for every homogeneous element F of S, $F^t \in S^{(t)}$.

Both Segre products and Veronese subrings arise naturally in projective geometry. Let $\mathrm{Proj}(R)$ denote the projective scheme associated with a standard graded K-algebra R. (This scheme is covered by open affines of the form $\mathrm{Spec}\,\big([R_F]_0\big)$, where F is a form of positive degree in R. To get an open cover it suffices to use finitely many F: any set of homgeneous generators F_j of an ideal primary to the homogeneous maximal ideal wil provide such a cover, and we may take the F_j to be one-forms.) An important reason for studying Segre products is that

$$\mathrm{Proj}(R \,\circledS_K\, S) \cong \mathrm{Proj}(R) \times \mathrm{Proj}(S).$$

The Veronese subrings of S have the property that

$$\mathrm{Proj}(S^{(t)}) = \mathrm{Proj}(S)$$

for all t. A specific homogeneous coordinate ring S for a projective scheme X over K (which means that $X = \mathrm{Proj}(S)$) gives an embedding of X in $\mathbb{P}^n_K$ by taking a degree-preserving mapping of a polynomial ring $K[X_0, \dots, X_n]$ onto R so as to give an isomorphism of vector spaces in degree 1. The Veronese subrings of R turn out to give a family of different embeddings of X into projective spaces. Although this is an important motivation for studying Veronese subrings, we shall not need to take this point of view in the sequel.

If M is any finitely generated $\mathbb{Z}$-graded S-module (there will be only finitely many nonzero negatively graded components), for every $i \in \mathbb{Z}_t$ we define

$$M_{i,t} = \bigoplus_{j \equiv i \bmod t\mathbb{Z}} M_j.$$

We then have that every $M_{i,t}$ is an $S^{(t)}$-module, and that

$$M = \bigoplus_{i \in \mathbb{Z}_t} M_{i,t}.$$

We may apply this notational convention to $M = S$ itself, to obtain a splitting of S into t modules over $S^{(t)}$. Then $S_{0,t} = S^{(t)}$.

We now want to consider the case where $S = K[X_1, \ldots, X_d]$ is a polynomial ring. In this case $S^{(t)}$ is generated over K by all monomials of degree t in $x_1, \ldots, x_d$. The elements $x_1^t, \ldots, x_d^t$ form a system of parameters in S and, hence, in $R = S^{(t)}$. The other generators of the the maximal ideal of R are integral over the ideal $(x_1^t, \ldots, x_d^t)S^{(t)}$, and so this parameter ideal is a minimal reduction of the maximal ideal of $S^{(t)}$. Note that any nonzero monomial of degree $t - i$, $0 \le i \le t-1$, multiplies $S_{i,t}$ into $S_{0,t} = S^{(t)}$. Therefore, every $S_{i,t}$ has rank one as an $S^{(t)}$-module.

It follows that the rank of S over $S^{(t)}$ is t, since S is the direct sum of t modules over $S^{(t)}$, each of which has torsion-free rank one.

Since $x_1^t, \ldots, x_d^t$ generates a minimal reduction of the maximal ideal of $S^{(t)}$ which is a parameter ideal, we have that $e(S^{(t)})$ is the torsion-free rank of $S^{(t)}$ over $B = K[x_1^t, \ldots, x_d^t]$. Clearly, the torsion-free rank of S over B is t^d, and we have just seen that the torsion free-rank of S over $R = S^{(t)}$ is t. It follows that

$$e(R) = t^d/t = t^{d-1}.$$

(In the case of composite extensions of domains, torsion-free rank multiplies: one may pass to the fraction fields, and then the torsion-free rank is the same as the degree of the corresponding field extension.)

We next want to classify all the graded Cohen-Macaulay modules over $S^{(t)}$ when S is the polynomial ring in two variables. We shall use this classification to show that there is a unique graded linear maximal Cohen-Macaulay module that is indecomposable, i.e., not a direct sum.

In the case of the polynomial ring in three variables, we shall, at least, exhibit a graded module that is a linear maximal Cohen-Macaulay module.