Math 711: Lecture of December 6, 2006

We can now analyze all graded maximal Cohen-Macaulay modules for a Veronese subring of the polynomial ring in two variables, and show, as a corollary, that there is a unique indecomposable linear maximal Cohen-Macaulay module up to shifts in grading. Recall that if M is a $\mathbb{Z}$-graded module and $h \in \mathbb{Z}$, $M(h)$ denotes the same module, graded so that $[M(h)]_n = [M]_{h+n}$ for all n. We also need:

Discussion: reflexive modules over normal domains of dimension 2. Let M and W be any R-modules. Then there is a natural canonical map

$$M \to \mathrm{Hom}_R\bigl(\mathrm{Hom}_R(M,\, W),\, W\bigr)$$

whose value on $u \in M$ is the map θ_u defined by

$$\theta_u(f) = f(u).$$

Recall that an R-module M is *reflexive* if the natural map

$$M \to \mathrm{Hom}_R\bigl(\mathrm{Hom}_R(M,\, R),\, R\bigr)$$

is an isomorphism.

Notice that if $x,\, y \in R$ form a regular sequence on W, they form a regular sequence on $V = \mathrm{Hom}_R(M,\, W)$ whenever $V \neq 0$. First note that if $f \in V$, then $xf = 0$ if and only if x kills all values of f, and this implies that $f = 0$, since x is not a zerodivisor W. Second, if $xf = yg$ then for all u in M, $xf(u) = yg(u)$, and this implies that $g(u)$ is, in a unique way, a multiple of x, i.e., there exists a unique element of W, which we may denote $h(u)$, such that $g(u) = xh(u)$. It is easy to check that h is an R-linear map from $M \to W$, and so $g = xh$. We have shown that $x,\, y$ is a regular sequence on $\mathrm{Hom}_R(M,\, W)$.

This helps explain the following fact, which is a particular case of the Theorem on p. 2 of the Lecture Note from Math 615, March 29, 2004.

Theorem. *A finitely generated module over a Noetherian normal domain of dimension two is maximal Cohen-Macaulay if and only if it is reflexive.*

We also note:

Lemma. *Let M be a finitely generated $\mathbb{Z}$-graded module over the polynomial ring $S = K[X_1,\, \dots\,, X_d]$. Then M has depth d on the maximal ideal of S if and only if M is S-free.*

Proof. This is a graded version of a special case of the Auslander-Buchsbaum theorem, but we give an elementary proof. The "if" part is obvious. Suppose that the depth is

d. Let $u_1, \dots, u_h$ be forms of M whose images in $M/(x_1, \dots, x_d)M$ form a K-basis for $M/(x_1, \dots, x_d)M$. It will suffice to show that $u_1, \dots, u_h$ is a free basis for M over S. The case where $d = 0$ is obvious, and we use induction on d. The depth condition implies that $x_1, \dots, x_d$ is a regular sequence on M, and the induction hypothesis implies that M/x_1M is free on the images of the u_j over $K[x_2, \dots, x_d]$. The homogeneous Nakayama Lemma implies that $u_1, \dots, u_h$ span M. We must show that there is no nonzero relation on the u_j. If there is a relation

$$\sum_{j=1}^{h} F_j u_j = 0$$

with some $F_j \neq 0$, by taking homogeneous components we may assume that $\deg F_j + \deg u_j$ is constant, say δ, and we may choose δ as small as possible for a nonzero homogeneous relation. Consider the relation modulo x_1S. By the induction hypothesis, it must vanish, so that every F_j can be written x_1G_j, and then we have

$$x_1(\sum_{j=1}^{h} G_j u_j = 0).$$

Since x_1 is not a zerodivisor on M, we have that

$$\sum_{j=1}^{h} G_j u_j = 0,$$

which gives a relation of lower degree, a contradiction. $\square$

Theorem. *Let M be a graded maximal Cohen-Macaulay module over $R = S^{(t)}$, where $S = K[X, Y]$ is a polynomial ring in two variables over a field K. Then M is a finite direct sum of modules $S(h)_{j,t}$, each of which is a maximal Cohen-Macaulay module.*

Proof. Let M be any maximal $\mathbb{Z}$-graded Cohen-Macaulay module over R. Since S is Cohen-Macaulay, it is a maximal Cohen-Macaulay R-module, and, hence, each of the modules $S_{j,t}$ is a maximal Cohen-Macaulay R-module.

Because $R \to S$ splits, we obtain an R-split embedding $M \hookrightarrow S \otimes_R M$ as R-modules. The $\mathbb{Z}_t$-indexed splitting of S as an R-module induces such a splitting on $S \otimes_R M$, where the degree 0 component is M. Then we have

$$\mathrm{Hom}_R\big(\mathrm{Hom}_R(M, R)\, R\big) \hookrightarrow \mathrm{Hom}_R\big(\mathrm{Hom}_R(S \otimes_R M, R), R\big)$$

where the module on the right continues to have both a graded S-modoule structure and a $\mathbb{Z}_t$-indexed splitting into R-modules. It has depth two as an R-module, since R does, and so it has depth two as a graded S-module. Thus, by the Lemma, the module is a finite direct sum of modules $S(h_\nu)$ with h_ν varying.

The module on the left is a split direct summand and is, in fact, the index 0 summand of the module on the right in the splitting indexed by $\mathbb{Z}_t$. However, since M is maximal Cohen-Macaulay and R is normal of dimension two, we have that

$$M \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(M,\, R)\, R\big)$$

is an isomorphism. The stated conclusion follows at once. $\square$

Corollary. *Let $R = S^{(t)}$, where $S = K[X,\, Y]$ is a polynomial ring in two variables over a field K. Then a graded R-module M is a linear maximal Cohen-Macaulay module over R if and only if M is a direct sum of copies of modules $S(h)_{t-1,t}$. Thus, M is an indecomposable linear maximal Cohen-Macaulay module if and only if it is, up to a shift in grading, $S_{t-1,t}$.*

Proof. A direct sum of modules is maximal Cohen-Macaulay if and only if each summand is, and both $\nu(\_)$ and $e(\_)$ are additive over direct sums. It follows that a direct sum of nonzero modules is a linear maximal Cohen-Macaulay module if and only if every summand is a linear maximal Cohen-Macaulay module. Since all of the $S_{j,t}$ are maximal Cohen-Macaulay modules of torsion-free rank one, each of them has multiplicity t. The result now follows because $S_{j,t}$ is minimally generated by the monomials of degree j, namely

$$X^j,\, X^{j-1}Y,\, ,\, \ldots\, ,\, XY^{j-1},\, Y^j,$$

in X and Y, and so $\nu(S_{j,t}) = j+1$, $0 \le j \le t-1$. Obviously, $S_{j,t}$ is a linear maximal Cohen-Macaulay module if and only if $j = t-1$. $\square$

We next want to show that when $S = K[X,\, Y,\, Z]$, the polynomial ring in three variables over the field K, one can construct linear maximal Cohen-Macaulay modules over $R = S^{(t)}$ for all $t \ge 1$. We first note:

Lemma. *Let A be an $r \times s$ matrix over an arbitrary ring R and let Q be the cokernel of the map $A : R^s \to R^r$; Q is also the quotient of R^r by the column space of A. Then $I_r(A)$ kills Q, i.e., $I_r(A)R^r \subseteq Im(A)$.*

Proof. Let D denote the determinant of an $r \times r$ minor of A. By permuting the columns, we might as well assume that D corresponds to the first r columns of A. It suffices to show that the product of D with every standard basis vector for R^r, written as a column, is in the column space of A, and so it certainly suffices to prove that it is in the R-span of the first r columns. Therefore, we might as well replace A by the submatrix formed from its first r columns. We change notation, so that A is now an $r \times r$ matrix. Let B denote the classical adjoint of A, which is the $r \times r$ matrix that is the transpose of the matrix of cofactors of A. Then $AB = D\boldsymbol{I}_r$. Since each column of AB is the product of A with a column of B, and since the columns of $D\boldsymbol{I}_r$ are precisely the products of D with the standard basis for R^r, the result follows. $\square$

We are now ready to construct a linear maximal Cohen-Macaulay module over $R = S^{(t)}$. To this end, let A denote the $t-1 \times t+1$ matrix

$$\begin{pmatrix} X & Y & Z & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & X & Y & Z & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & X & Y & Z & \cdots & 0 & 0 & 0 \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & X & Y & Z \end{pmatrix}$$

where the i th row has entries X, Y, and Z in the i th, $i+1$ st, and $i+2$ nd spots, respectively, and 0 everywhere else, $1 \le i \le t-1$. We have an exact sequence:

$$(*) \quad 0 \to N \to S(-1)^{\oplus t+1} \xrightarrow{A} S^{\oplus t-1}.$$

Theorem. *Let notation be as above, so that $S = K[X, Y, Z]$ is the polynomial ring in three variables over a field K, $R = S^{(t)}$ for a positive integer t, and $N \subseteq S(-1)^{\oplus t+1}$ is the kernel of the matrix A defined above. Then the module $M = N_{t-1,t} \subseteq N$ is a linear maximal Cohen-Macaulay module over R of torsion-free rank 2, with minimal generators all of the same degree.*

Proof. Using the splitting indexed by $\mathbb{Z}_t$, the sequence $(*)$ displayed above yields

$$(**) \quad 0 \to M \to S_{t-2,t}^{\oplus t+1} \xrightarrow{A} S_{t-1,t}^{\oplus t-1}$$

where the map on the right is the restriction of the linear map with matrix A. By the Lemma above, the image of $A : S(-1)^{\oplus t+1} \to S^{\oplus t-1}$ contains $I_{r-1}(A)S^{\oplus t-1}$. By Problem 2. of Problem Set #5,

$$I_{r-1}(A) = (X, Y, Z)^{t-1}S.$$

But $[(X, Y, Z)^{t-1}S]_{t-1,t} = S_{t-1,t}$, and it follows that the restricted map induced by A in $(**)$ is surjective, i.e., that

$$0 \to M \to S_{t-2,t}^{\oplus t+1} \xrightarrow{A} S_{t-1,t}^{\oplus t-1} \to 0$$

is exact. Since the modules in the middle and on the right are maximal Cohen-Macaulay modules, so is M. Since the rank of every $S_{j,t}$ is one, the module in the middle has rank $t+1$, and the module on the right has rank $t-1$. It follows that M has rank 2, and so $e(M) = 2\,e(R) = 2t^2$.

To complete the proof, it will suffice to show that $\nu(M) = 2t^2$ as well. If we think of $M \subseteq S_{t-2,t}^{\oplus t+1}$, the least degree (using degree in S for every component) in which there might be nonzero elements of M is $t-2$. Now,

$$\dim([S]_n = \binom{n+2}{2},$$

and so the dimension of the piece of M that lies in $S_{t-2,t}^{\oplus t+1}$ is

$$(t+1)\binom{t}{2} - (t-1)\binom{t+1}{2} = \frac{(t+1)t(t-1)}{2} - \frac{(t-1)(t+1)t}{2} = 0$$

The next possible degree in which M might be nonzero is $t+t-2 = 2t-2$, and here we get

$$(t+1)\binom{2t}{2} - (t-1)\binom{2t+1}{2} = \frac{(t+1)(2t)(2t-1)}{2} - \frac{(t-1)(2t+1)2t}{2} = 2t^2.$$

Clearly, one needs $2t^2$ minimal generators in this degree, and these elements must generate, since $\nu(M) \le e(M)$ always.

We give an alternative argument. First note that if $y_1, \dots, y_h$ is a regular sequence on all of the modules in the short exact sequence

$$(\#) \quad 0 \to M \to M' \to M'' \to 0$$

then it is easy to see by induction on h that

$$(\#\#) \quad 0 \to M/(y_1, \dots, y_h)M \to M'/(y_1, \dots, y_h)M' \to M''/(y_1, \dots, y_h)M'' \to 0$$

is exact, and since the short exact sequence (#) maps onto the short exact sequence (##) the nine lemma implies that the sequence of kernels

$$0 \to (y_1, \dots, y_h)M \to (y_1, \dots, y_h)M' \to (y_1, \dots, y_h)M'' \to 0$$

is exact as well.

We know, as in the first argument, know that there are no elements of $M \subseteq S_{t-2,t}^{\oplus t+1}$ in degree $t-2$. Every element of M, thought of a submodule of $S^{\oplus t+1}$, has degree $2t-2$ or more. If m is the maximal ideal of R, which is generated by the monomials of degree t in X, Y, Z, we have that all elements of mM have degree $3t-2$ or greater, and every monomial of degree $3t-2$ or more in X, Y, Z must involve X^t or Y^t or Z^t. Hence, $mM \subseteq (X^t, Y^t, Z^t)S^{\oplus t+1}$, and it follows that $mM \subseteq (X^t, Y^t, Z^t)S_{t-2,t}^{\oplus t+1}$. Since all three of the modules M, $S_{t-2,t}^{\oplus t+1}$, and $S_{t-1,t}^{\oplus t-1}$ are maximal Cohen-Macaulay modules over the ring R, we have that $X^t, Y^t, Z^t \in R$ is a regular sequence on all of them, and so we see that

$$0 \to (X^t, Y^t, Z^t)M \to (X^t, Y^t, Z^t)S_{t-2,t}^{\oplus t+1} \to (X^t, Y^t, Z^t)S_{t-1,t}^{\oplus t-1} \to 0$$

is exact. It follows that $mM \subseteq (X^t, Y^t, Z^t)M$, and so they are equal, which is what we need for M to be linear. $\square$