Math 711, Fall 2006 **Problem Set #1**
Due: Monday, October 2

1. $R$ is a ring and $W$ is a multiplicative system in $R$.

(a) Suppose that $R \subseteq S$ is a ring extension. Let $R'$ be the integral closure of $R$ in $S$. Show that the integral closure of $W^{-1}R$ in $W^{-1}S$ is $W^{-1}R'$.

(b) Let $I$ be an ideal of $R$. Show that $\overline{IW^{-1}R} = \overline{I}W^{-1}R$.

2. Let $R$ be a reduced ring with finitely many minimal primes $P_1, \ldots, P_n$. For $1 \leq i \leq n$, let $D_i$ be $R/P_i$, let $L_i$ be the fraction field of $D_i$, and let $D_i'$ be the integral closure of $D_i$ in $L_i$. Show that the total quotient ring of $R$ is isomorphic with $T = \prod_{i=1}^{n} L_i$ and that the integral closure of $R$ in $T$ is isomorphic with $\prod_{i=1}^{n} D_i'$. (Note that this always applies when $R$ is a reduced Noetherian ring.)

3. Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$. If $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let $\underline{x}^{\underline{a}}$ denote $x_1^{a_1} \cdots x_n^{a_n}$. The ideals $I$ of $R$ generated by monomials correspond bijectively to the subsets $H \subseteq \mathbb{N}^n$ with the property that if $\underline{a} \in H$ and $\underline{b} \geq \underline{a}$ in the sense that $b_i \geq a_i$ for all $i$, then $\underline{b} \in H$. Sets $H \subseteq \mathbb{N}^n$ with this property are called *semigroup ideals* of $\mathbb{N}^n$. Under the bijection, $I$ is the $K$-vector space span of $\{\underline{x}^{\underline{a}} : \underline{a} \in H\}$, and $H$ is the set $\{\underline{a} \in \mathbb{N} : \underline{x}^{\underline{a}} \in I\}$. (You may assume this bijection.) Show that the integral closure of the ideal corresponding to $H$ is a monomial ideal, and corresponds to $H'$, where $H'$ is the intersection of $\mathbb{N}^n$ with the convex hull of $H$ over the rational numbers $\mathbb{Q}$.

4. Let $K$ be a finite field with $q$ elements, let $T = K[[X, Y]]$, a formal power series ring in two variables over $K$, and let $f \in T$ have leading (i.e., *lowest* degree) form equal to $XY(X^{q-1} - Y^{q-1})$ (which is divisible by all forms of degree one in $K[X, Y]$). Let $R = K[[X, Y]]/(f) = K[[x, y]]$. Show that the analytic spread of the maximal ideal $m$ of $R$ is 1, but that $m$ has no reduction with just one generator.

5. Let $(R, m, K)$ be a local domain. Show that there is a DVR $(V, m_V)$ such that $R \subseteq V$ and $m \subseteq m_V$. (Show that $R$ embeds in $\widehat{R}/P = S$, where $P$ is minimal, and this ring is module-finite over a complete regular local ring $A$. Solve the problem for $A$ and complete the DVR to get, say, $W$. Normalize $W[S]$).)

6. Let $(R, m, K)$ be a local domain and let $I \subseteq m$ be an ideal.

Suppose that $x \in R$ is not in the integral closure of $I$.

(a) Consider the ring $R[I/x]$ generated by all fractions with numerator in $I$ and denominator $x$. Show that there is a maximal ideal $Q$ of this ring containing $m$ and all the elements of $I/x$, and let $(T, n)$ be the localization of this ring at $Q$. Then $m$ and all elements of $I/x = \{i/x : i \in I\}$ are in $n$.

(b) By **#5.** there is a DVR $V$ with $T \subseteq V$ whose valuation $v$ is nonnegative on $T$ and positive on $n$. Show that $v$ is positive on $m$ but takes a smaller value on $x$ than on any element of $I$.

Conclude from the statements above that the integral closure of $I$ is an intersection of $m$-primary integrally closed ideals.