Math 711, Fall 2006 **Problem Set #1 Solutions**

1. (a) The image of an equation for integral dependence of an element of R' is an equation showing integral dependence for its image in $W^{-1}S$. This implies that the integral closure T contains $W^{-1}R'$. Now suppose that we have $t^d + (r_1/w_1)t^{d-1} + \cdots + (r_d/w_d) = 0$ with t in $W^{-1}S$. Let w be divisible by all the w_i be such that $wt \in S$ and multiply by w^d. This yields $(wt)^d + r_1'(wt)^{d-1} + \cdots r_d' \in S$ and this element is 0 in $W^{-1}S$, so that it is killed by $w' \in W$. We can now multiply the equation by ${w'}^t$ to get an equation of integral dependence for $ww't$ over R. Thus, $ww't \in R'$, and $t \in W^{-1}S$. $\square$

(b) The ring $W^{-1}R[W^I t]$ may be identified with $W^{-1}R[It]$, and so its integral closure in $W^{-1}R[t]$ may be identified with $W^{-1}(R + \overline{I}t + \overline{I^2}t^2 + \cdots)$. This yields that the integral closure of $W^{-1}I$ in $W^{-1}R$ may be identified with $W^{-1}\overline{I}$. $\square$

2. If W is the multipicative system of nonzerodivisors, $W^{-1}R$ is a 0-dimensional reduced ring with finitely many prime ideals: these are maximal, and by the Chinese Remainder Theorem it is isomorphic with the product of the quotients by these primes, which is the product of the L_i. An element of this product is integral over R iff each of it components is: the condition is obviously necessary, and it is sufficient, for if the i th component satisfies a monic polynomial f_i for all i, the element is a root of the product of the f_i.

3. In a $\mathbb{Z}$- or $\mathbb{N}$-graded ring R, the integral closure of a homogeneous ideal I is homogenous: the grading extends to $R[It] \subseteq R[t]$ (with t having degree 0). The integral closure of the ring in $RI[t]$ is generated in t-degree 1 by elements ft with $f \in \overline{I}t$: the homogeneous components of every ft are therefore in $\overline{I}t$, and the result follows.

R may be viewed as $B_i[x_i]$ where B is the polynomial ring over K in the variables other than x_i. This provides an $\mathbb{N}$-grading for each i. The generators of the ideal are homogeneous in each of these gradings. Given an element of $\overline{I}$ each of its components of a fixed degree in x_1 is therefore in $\overline{I}$. Repeating this argument for x_2, x_3, and, eventually, x_n, we see that $\overline{I}$ is generated by monomials. Consider an equation of integral dependence of degree h for a monomial μ on I. By taking homogenous components for the $\mathbb{N}^n$-grading on $K[x_1, \ldots, x_n]$, we may assume that the equation is $\mathbb{N}^n$-homogeneous. By dividing by an appropriate power of the indeterminate, we may assume that the constant term is not 0. Up to scalar multiplication, the constant term is a monomial in I^h, which must be the same as μ^h. Thus, we must have $\mu^h = \mu_1^{k_1} \cdots \mu_s^{k_s}$ with the $\mu_j \in I$ and $\sum_{j=1}^s k_j = h$. If we consider the corresponding vectors $\beta, \alpha_1, \ldots, \alpha_s$, we get $h\beta = \sum_{j=1}^s k_j\alpha_j$ or $\beta = \sum_{j=1}^s (k_j/h)\alpha_j$, where the coefficients are nonnegative rationals whose sum is 1, as required. Conversely, given such an equation, we can clear denomiators to get one of the form $h\beta = \sum_{j=1}^s k_j\alpha_j$ where $h > 0$, $k_j \geq 0$ are integers and $h = \sum_{j=1}^s k_j$. We can then conclude that $\mu^h \in I^h$. $\square$

4. The analytic spread is at least the height of m and at most the dimension of R, both of which are 1. Therefore, it is one. The associated graded ring may be identified with $K[X, Y]/(F)$, where $F = XY(X^{q-1} - Y^{q-1})$. There is a reduction with just one generator iff there is a form of degree 1 generating an ideal primary to the homogeneous maximal ideal. But all degree 1 forms divide F and so are zerodivisors in $\mathrm{gr}_m(R)$. $\square$

5. Since $T = \widehat{R}$ is faithfully flat over R, no element of $R - 0$ is a zerodivisor in T, and any minimal prime of T meets R in (0). Hence, R embeds in a complete domain T/P. If A is regular, we can define a valuation by letting the order of $a \neq 0$ be the highest power of the maximal ideal M of A to which a belongs. (The order of ab is the sum of the orders because $\mathrm{gr}_M A$ is a polynomial ring.) This gives an embedding of A into a DVR, which we may complete: call it W. Then $W[S]$ is a module-finite extension domain of the complete domain W, and so is a complete local domain of dimension one. Its normalization W' is module-finite over it, and is again local, so that it is a normal complete local domain of dimension one. Then W' is a DVR, and $R \subseteq S \subseteq W'$. $\square$

6. (a) $(m, I/x)R[I/x] = m + I/x + I^2/x^2 + \cdots + I^k/x^k + \cdots$, where any single element is a finite sum. Thus, we must show that we cannot have $1 = u + i_1/x + i_2/x^2 + \cdots + i_k/x^k$ with $i_j \in I^j$, $1 \leq j \leq k$. If we have this, multiply by x^k to get $(1-u)x^k - i_1x^{k-1} - \cdots - i_k = 0$. Since $u \in m$, $1 - u$ is a unit and we may multiply by its inverse to get an equation of integral dependence for x on I, a contradiction. $\square$

(b) v is positive on m since $m \subseteq n$ by construction. Likewise, for all $i \in I$, $i/x \in n$ so that $v(i) - v(x)$ is positive. IV is integrally closed since it is principal in V, and V is normal. The contraction of an integrally closed ideal is integrally closed, and so its contraction $\mathfrak{A}$ to R is integrally closed. $\mathfrak{A}$ is clearly m-primary, and contains I but not x.

Thus, every $x \notin \overline{I}$ is also not contained in an m-primary integrally closed ideal $\mathfrak{A} \supseteq I$, and it follows at once that $\overline{I}$ is the intersection of all such $\mathfrak{A}$. $\square$