Math 711, Fall 2006 **Problem Set #2**
Due: Friday, October 20

1. Let $S = \mathrm{gr}_I(R)$ where $(R,\, m,\, K)$ is local and $I \subseteq m$. Let $\mathcal{M}$ denote the homogeneous maximal ideal $m/I + I/I^2 + I^2/I^3 + \cdots + I^k/I^{k+1} + \cdots \subseteq \mathrm{gr}_I(R)$. Show that if $S_{\mathcal{M}}$ is an integral domain, then S is an integral domain, and R is an integral domain.

2. Suppose that I is an ideal of a Noetherian domain R and that $\mathrm{gr}_I R$ is an integral domain. For each $r \in R - \{0\}$, let $v(r)$ be the unique integer $n \geq 0$ such that $r \in I^n - I^{n+1}$.

(a) Prove that v is a valuation, i.e., that if $r,\, r' \in R - \{0\}$, $v(rr') = v(r) + v(r')$, and $v(r + r') \geq \min\{v(r),\, v(r')\}$ with equality if $v(r) \neq v(r')$.

(b) Note that v extends to $\mathcal{K} = \mathrm{frac}\,(R)$ (assume this). Let V be the DVR of $\mathcal{K}$ associated with v, i.e., $V = \{u \in \mathcal{K} - \{0\} : v(u) \geq 0\} \cup \{0\}$. Prove that IV is the maximal ideal of V, and that for all $n \geq 0$, $I^nV \cap R = I^n$. Deduce that I^n is integrally closed for all n.

3. (a) Let R be a Noetherian domain and let P be prime. Let $P^{(n)}$ denote the contraction of P^nR_P to R, i.e. the *n th symbolic power* of P. Suppose that t is an indeterminate over R and that the *symbolic power algebra* $\bigoplus_{n=0}^{\infty} P^{(n)}t^n \subseteq R[t]$ is finitely generated over R. Prove that for some integer $k \geq 1$, $P^{(nk)} = (P^{(k)})^n$ for every positive integer n.

(b) Suppose also that R is a 2-dimensional normal domain, m is a maximal ideal, and that $K = R/m$ is infinite. Assume that R is local, or else that R is a finitely generated $\mathbb{N}$-graded algebra such that $R_0 \hookrightarrow R \twoheadrightarrow K$ is an isomorphism, and that m is the homogeneous maximal ideal. Let P be a height one prime of R, homogeneous in the graded case. Prove that for k as in part (a), $P^{(k)}$ is principal. (Suggestion: first prove that $\mathfrak{an}\big(P^{(k)}\big) = 1$.) Thus, if P has infinite order in the divisor class group $\mathcal{C}\ell\,(R)$, the symbolic power algebra, which may be written as $R[Pt]_P \cap R[t]$, is *not* Noetherian.

4. Let R be a Noetherian domain and b a nonzero element such that R_b is normal.

(a) Prove that R is normal if and only if R_P is normal for every associated prime of b.

(b) Prove that $\{P \in \mathrm{Spec}\,(R) : R_P \text{ is not normal}\}$ is the union of the sets $V(P)$, where P is an associated prime of P such that R_P is not normal, and so is Zariski closed.

(c) Prove that if R_m has module-finite integral closure for every maximal ideal m of R, then R has module-finite integral closure. (For every module-finite extension T of R within its fraction field, $T_b = R_b$, and the image of the non-normal locus of T in $\mathrm{Spec}\,(R)$ is closed. Choose T to make this locus minimal, and then show that the locus is empty.)

5. Let $(R,\, m,\, K)$ be a complete local reduced ring and I an ideal of R. Show that the integral closure of $R[It]$ is module-finite over $R[It]$. (If not, consider an infinite increasing sequence of graded module-finite extensions S_i within the total quotieont ring. Show that the sequence persists when one passes to $R[[It]] \subseteq R[[t]]$, which may be identified with the completion of $R[It]$ localized at the maximal ideal $\mathcal{M} = mR[It] + ItR[It]$. Use the fact that the integral closure of the reduced complete ring $R[[It]]$ is module-finite over it to get a contradiction.)

6. Let R be a normal domain and I an ideal. Suppose that the normalization S of $R[It]$ is module-finite over $R[It]$. Prove that the normalization of $R[I/f]$ is module-finite over $R[I/f]$ for every $f \in I - \{0\}$. (Suggestion: consider $[R[It]_{ft}]_0$ and $[S_{ft}]_0$.)